

POLYNOMIAL INTERPOLATION OVER QUATERNIONS

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ABSTRACT. Interpolation theory for complex polynomials is well understood. In the non-commutative quaternionic setting, the polynomials can be evaluated “on the left” and “on the right”. If the interpolation problem involves interpolation conditions of the same (left or right) type, the results are very much similar to the complex case: a consistent problem has a unique solution of a low degree (less than the number of interpolation conditions imposed), and the solution set of the homogeneous problem is an ideal in the ring $\mathbb{H}[z]$. The problem containing both “left” and “right” interpolation conditions is quite different: there may exist infinitely many low-degree solutions and the solution set of the homogeneous problem is a quasi-ideal in $\mathbb{H}[z]$.

1. INTRODUCTION

Given distinct points $z_1, \dots, z_n \in \mathbb{C}$ and target values $c_1, \dots, c_n \in \mathbb{C}$, the Lagrange interpolation problem consists of finding a complex polynomial $f \in \mathbb{C}[z]$ such that

$$f(z_i) = c_i \quad \text{for } i = 1, \dots, n. \quad (1.1)$$

A particular solution to this problem is the *Lagrange interpolation polynomial*

$$\tilde{f}(z) = \sum_{j=1}^n \frac{c_j p_j(z)}{p_j(z_j)}, \quad \text{where } p_j(z) = \prod_{\substack{i=1 \\ i \neq j}}^n (z - z_i), \quad (1.2)$$

while all polynomials satisfying conditions (1.1) are parametrized by the formula

$$f(z) = \tilde{f}(z) + p(z)h(z), \quad p(z) = \prod_{i=1}^n (z - z_i), \quad h \in \mathbb{C}[z], \quad (1.3)$$

where h is the free parameter. The latter representation holds since the mapping $f \mapsto (f(z_1), \dots, f(z_n))$ is linear from $\mathbb{C}[z]$ to \mathbb{C}^n and since the solution set of the corresponding homogeneous problem is the ideal in $\mathbb{C}[z]$ generated by p .

Over the years, Lagrange interpolation has been played a prominent role in approximation theory and numerical analysis; more recent applications include image processing and control theory. The problem can be settled exactly as in the complex case for polynomials over any field (including finite fields, which has applications in cryptography). However, interpolation problems in non-commutative polynomial rings have not attracted much attention so far. The objective of this paper is to study the Lagrange interpolation problem for polynomials over the skew field \mathbb{H} of real quaternions

$$\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \quad (x_0, x_1, x_2, x_3 \in \mathbb{R}), \quad (1.4)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are imaginary units commuting with the reals and such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. We denote by $\mathbb{H}[z]$ the ring of polynomials in one formal variable z which commutes with quaternionic coefficients. The ring operations in $\mathbb{H}[z]$ are defined as in the commutative case, but as multiplication in \mathbb{H} is not commutative, multiplication in $\mathbb{H}[z]$ is not commutative either. For $\alpha \in \mathbb{H}$ and $f \in \mathbb{H}[z]$, we define $f^{e\ell}(\alpha)$ and $f^{er}(\alpha)$ (left and right evaluations of f at α) by

$$f^{e\ell}(\alpha) = \sum_{k=0}^n \alpha^k f_k, \quad f^{er}(\alpha) = \sum_{k=0}^n f_k \alpha^k, \quad \text{if } f(z) = \sum_{k=0}^n z^k f_k, \quad f_k \in \mathbb{H}. \quad (1.5)$$

Since \mathbb{R} is the center of \mathbb{H} , the ring $\mathbb{R}[z]$ of polynomials with real coefficients is the center of $\mathbb{H}[z]$. As a consequence of this observation, we mention two cases where the left and the right evaluations produce the same result.

Remark 1.1. *If $x \in \mathbb{R}$, then $f^{e\ell}(x) = f^{er}(x)$ for every $f \in \mathbb{H}[z]$. On the other hand, if $f \in \mathbb{R}[z]$, then $f^{e\ell}(\alpha) = f^{er}(\alpha)$ for every $\alpha \in \mathbb{H}$.*

In general, interpolation conditions imposed by the left and the right evaluations should be distinguished. We will consider the interpolation problem whose data set consists of two (not necessarily disjoint) finite sets

$$\Lambda = \{\alpha_1, \dots, \alpha_n\} \quad \text{and} \quad \Omega = \{\beta_1, \dots, \beta_m\} \quad (1.6)$$

of distinct elements in \mathbb{H} along with the respective target values c_1, \dots, c_n and d_1, \dots, d_m in \mathbb{H} . The *two-sided Lagrange problem* consists of finding an $f \in \mathbb{H}[z]$ such that

$$f^{e\ell}(\alpha_i) = c_i \quad \text{for } i = 1, \dots, n, \quad (1.7)$$

$$f^{er}(\beta_j) = d_j \quad \text{for } j = 1, \dots, m. \quad (1.8)$$

The problem will be termed *left-sided* if $\Omega = \emptyset$ and *right-sided* if $\Lambda = \emptyset$. Since right and left evaluations coincide at real points, we may assign all real interpolation nodes to the left set Λ assuming therefore, that $\Omega \cap \mathbb{R} = \emptyset$. We emphasize that the sets (1.6) do not have to be disjoint, so that we may have left and right interpolation conditions at the same interpolation node $\alpha_i = \beta_j$. One-sided interpolation problems in quaternionic and related non-commutative settings were previously discussed in [7, 5, 3, 10], mostly due to their connections with quaternionic Vandermonde matrices. These results are recalled in Section 3.1 below.

The paper is organized as follows. Section 2 contains the background on quaternionic polynomials and their (left and right) zeros; left and right minimal polynomials are also recalled in Section 2. In Section 3 we present certain necessary conditions for the problem to have a solution and show that in case these conditions are met, one can assume without loss of generality that none three elements in the set $\Lambda \cup \Omega$ belong to the same conjugacy class. It is shown in Section 4 that if some elements in Λ have conjugates in Ω , there are additional necessary (and this time, sufficient) conditions for the problem to be solvable given in terms of certain Sylvester equations and related to certain backward-shift operators on $\mathbb{H}[z]$. In Section 5 we present the

parametrization of all solutions $f \in \mathbb{H}[z]$ to the two-sided problem (1.7), (1.8) in the form

$$f(z) = \tilde{f}(z) + f_h(z; \mu_1, \dots, \mu_k) + P_{\Lambda, \ell}(z) \cdot h(z) \cdot P_{\Omega, r}(z), \quad (1.9)$$

where \tilde{f} is a particular low-degree solution, f_h is the general low-degree solution of the related homogeneous problem containing free parameters μ_1, \dots, μ_k (each parameter is associated to a pair $(\alpha_i, \beta_j) \in \Lambda \times \Omega$ of equivalent interpolation nodes and varies in a two-dimensional real subspace of \mathbb{H}), $P_{\Lambda, \ell}$ and $P_{\Omega, r}$ are respectively the left and the right minimal polynomials of the sets Λ and Ω , and h is the free parameter in $\mathbb{H}[z]$. The parametrization formula (1.9) somewhat resembles well known results on bi-tangential interpolation for matrix-valued complex polynomials [2, 1], in particular, the fact that in case of non-empty intersection of Λ and Ω , an additional condition is needed to guarantee the uniqueness of a low-degree solution.

2. BACKGROUND

To start, let us fix notation and terminology. For $\alpha \in \mathbb{H}$ of the form (1.4), its real and imaginary parts, the quaternion conjugate and the absolute value are defined as $\text{Re}(\alpha) = x_0$, $\text{Im}(\alpha) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, $\bar{\alpha} = \text{Re}(\alpha) - \text{Im}(\alpha)$ and $|\alpha|^2 = \alpha\bar{\alpha} = |\text{Re}(\alpha)|^2 + |\text{Im}(\alpha)|^2$, respectively. As in the complex case, $\alpha + \bar{\alpha} = 2\text{Re}(\alpha)$.

Definition 2.1. Two quaternions α and β are called *equivalent* (conjugate to each other) if $\alpha = h^{-1}\beta h$ for some nonzero $h \in \mathbb{H}$.

It follows (see e.g., [12]) that

$$\alpha \sim \beta \quad \text{if and only if} \quad \text{Re}(\alpha) = \text{Re}(\beta) \text{ and } |\alpha| = |\beta|. \quad (2.1)$$

Therefore, the conjugacy class of a given $\alpha \in \mathbb{H}$ form a 2-sphere (of radius $|\text{Im}(\alpha)|$ around $\text{Re}(\alpha)$) which will be denoted by $[\alpha]$. It is clear that $[\alpha] = \{\alpha\}$ if and only if $\alpha \in \mathbb{R}$.

2.1. Polynomial conjugation. The conjugate of a polynomial f is defined by

$$f^\sharp(z) = \sum_{j=0}^n z^j \bar{f}_j \quad \text{if} \quad f(z) = \sum_{j=0}^n z^j f_j. \quad (2.2)$$

The anti-linear involution $f \mapsto f^\sharp$ can be viewed as an extension of the quaternionic conjugation $\alpha \mapsto \bar{\alpha}$ from \mathbb{H} to $\mathbb{H}[z]$. A polynomial f is real if and only if $f \equiv f^\sharp$. Some further properties of polynomial conjugation are listed below:

$$ff^\sharp = f^\sharp f, \quad (fg)^\sharp = g^\sharp f^\sharp, \quad (fg)(fg)^\sharp = f(gg^\sharp)f^\sharp = (ff^\sharp)(gg^\sharp). \quad (2.3)$$

2.2. Left and right zeros of polynomials. We now recall some results on quaternionic polynomials and their roots needed for the subsequent analysis and presented here in the form suitable for our needs. For a more detailed exposition, we refer to [8] and references therein.

Definition 2.2. An element $\alpha \in \mathbb{H}$ is a *left zero* of $f \in \mathbb{H}[z]$ if $f^{e\ell}(\alpha) = 0$, and it is a *right zero* of f if $f^{er}(\alpha) = 0$.

We will denote by $\mathcal{Z}_\ell(f)$ and $\mathcal{Z}_r(f)$ the sets of all left and all right zeros of f respectively. It follows from Remark 1.1 that for a real polynomial $f \in \mathbb{R}[z]$, these two sets coincide: $\mathcal{Z}(f) := \mathcal{Z}_\ell(f) = \mathcal{Z}_r(f)$.

Example 2.3. For a non-real quaternion α , the real polynomial

$$\mathcal{X}_{[\alpha]}(z) = (z - \alpha)(z - \bar{\alpha}) = (z - \bar{\alpha})(z - \alpha) = z^2 - 2z \cdot \operatorname{Re}(\alpha) + |\alpha|^2 \quad (2.4)$$

is called the *characteristic polynomial* of the conjugacy class $[\alpha]$ (by (2.1), $\mathcal{X}_{[\alpha]} = \mathcal{X}_{[\beta]}$ if and only if $\alpha \sim \beta$) and can be characterized as a unique monic quadratic polynomial with the zero set equal $[\alpha]$. It is irreducible over \mathbb{R} since $\operatorname{Re}(\alpha) < |\alpha|$ and it is easily verified that conversely, any monic real quadratic polynomial without real roots is the characteristic polynomial of a unique quaternionic conjugacy class.

If $f \in \mathbb{R}[z]$, then for each $\alpha \in \mathbb{H}$ and $h \neq 0$, we have $f(h^{-1}\alpha h) = h^{-1}f(\alpha)h$ so that $\mathcal{Z}(f)$ contains, along with each α , the whole conjugacy class $[\alpha]$. For any $f \in \mathbb{H}[z]$, the polynomial $ff^\#$ is real and therefore, $\mathcal{Z}(ff^\#)$ is the union of finitely many conjugacy classes. The following result is due I. Niven [9].

Theorem 2.4. *Let $\deg(f) \geq 1$ and let $\mathcal{Z}(ff^\#) = \bigcup V_i$ be the union of distinct conjugacy classes. Then $\mathcal{Z}_\ell(f) \cup \mathcal{Z}_r(f) \subset \mathcal{Z}(ff^\#)$ and each conjugacy class contains at least one left and at least one right zero of f .*

Since any real polynomial of positive degree has a complex root, the latter theorem implies that for any $f \in \mathbb{H}[z]$ of positive degree, the zero sets $\mathcal{Z}_\ell(f)$ and $\mathcal{Z}_r(f)$ are not empty (the Fundamental Theorem of Algebra in $\mathbb{H}[z]$).

Remark 2.5. *As a consequence of the Euclidean algorithm which holds in $\mathbb{H}[z]$ in both left and right versions, we have*

$$\begin{aligned} \alpha \in \mathcal{Z}_\ell(f) &\iff f(z) = (z - \alpha)h(z) \quad \text{for some } h \in \mathbb{H}[z]; \\ \alpha \in \mathcal{Z}_r(f) &\iff f(z) = \tilde{h}(z)(z - \alpha) \quad \text{for some } \tilde{h} \in \mathbb{H}[z]. \end{aligned} \quad (2.5)$$

It follows from (2.5), that if $g^{e_\ell}(\alpha) = 0$, then $(gf)^{e_\ell}(\alpha) = 0$ for any $f \in \mathbb{H}[z]$. On the other hand, since $(gf)(z) = \sum_{k=0}^n z^k g(z) f_k$, we also have

$$(gf)^{e_\ell}(\alpha) = g^{e_\ell}(\alpha) \sum_{k=0}^n (g^{e_\ell}(\alpha)^{-1} \alpha g^{e_\ell}(\alpha))^k f_k = g^{e_\ell}(\alpha) f^{e_\ell}(g^{e_\ell}(\alpha)^{-1} \alpha g^{e_\ell}(\alpha)),$$

provided $g^{e_\ell}(\alpha) \neq 0$. Therefore, the left evaluation of the product of two polynomials is defined by the formula

$$(gf)^{e_\ell}(\alpha) = \begin{cases} g^{e_\ell}(\alpha) \cdot f^{e_\ell}(g^{e_\ell}(\alpha)^{-1} \alpha g^{e_\ell}(\alpha)) & \text{if } g^{e_\ell}(\alpha) \neq 0, \\ 0 & \text{if } g^{e_\ell}(\alpha) = 0. \end{cases} \quad (2.6)$$

Similarly, the right evaluation of the product is given by

$$(gf)^{e_r}(\alpha) = \begin{cases} g^{e_r}(f^{e_r}(\alpha) \alpha f^{e_r}(\alpha)^{-1}) \cdot f^{e_r}(\alpha) & \text{if } f^{e_r}(\alpha) \neq 0, \\ 0 & \text{if } f^{e_r}(\alpha) = 0. \end{cases} \quad (2.7)$$

Note that in case $\alpha \in \mathbb{R}$, both (2.6) and (2.7) simplify to $(gf)(\alpha) = g(\alpha)f(\alpha)$.

Lemma 2.6. *Let $f \in \mathbb{H}[z]$ and let $\alpha, \beta \in \mathbb{H}$ be two distinct conjugates: $\beta \in [\alpha] \setminus \{\alpha\}$. The following are equivalent:*

- (1) $\alpha, \beta \in Z_{\ell}(f)$; (2) $\alpha, \beta \in Z_{\mathbf{r}}(f)$; (3) $[\alpha] \subset Z_{\ell}(f) \cap Z_{\mathbf{r}}(f)$;
- (4) f can be factored as

$$f(z) = \mathcal{X}_{[\alpha]}(z)g(z) = g(z)\mathcal{X}_{[\alpha]}(z) \quad \text{for some } g \in \mathbb{H}[z]. \quad (2.8)$$

Proof. Implications (4) \Rightarrow (3) \Rightarrow (2) and (3) \Rightarrow (1) are trivial. To confirm (1) \Rightarrow (4), let us assume that $\alpha, \beta \in Z_{\ell}(f)$. Since $\beta \in [\alpha]$, then

$$\beta^2 - \beta(\alpha + \bar{\alpha}) + |\alpha|^2 = \mathcal{X}_{[\alpha]}(\beta) = 0$$

so that $\beta(\beta - \alpha) = \beta^2 - \beta\alpha = \beta\bar{\alpha} - |\alpha|^2 = (\beta - \alpha)\bar{\alpha}$, and we conclude:

$$(\beta - \alpha)^{-1}\beta(\beta - \alpha) = \bar{\alpha}, \quad \text{whenever } \beta \in [\alpha] \setminus \{\alpha\}. \quad (2.9)$$

Since $f^{e\ell}(\alpha) = 0$, f can be factored as in (2.5). We use the latter factorization to left-evaluate f at β ; according to (2.6) and (2.9),

$$f^{e\ell}(\beta) = (\beta - \alpha) \cdot h^{e\ell}((\beta - \alpha)^{-1}\beta(\beta - \alpha)) = (\beta - \alpha) \cdot h^{e\ell}(\bar{\alpha}). \quad (2.10)$$

Since $f^{e\ell}(\beta) = 0$ and since \mathbb{H} is a division ring, we conclude from (2.10) that $h^{e\ell}(\bar{\alpha}) = 0$. Then again by (2.5), h can be factored as $h(z) = (z - \bar{\alpha}) \cdot g(z)$ for some $g \in \mathbb{H}[z]$ which being combined with factorization (2.5) for f gives (2.9): $f(z) = (z - \alpha) \cdot h(z) = (z - \alpha)(z - \bar{\alpha}) \cdot g(z) = \mathcal{X}_{[\alpha]}(z) \cdot g(z)$. \square

The last lemma supplements Theorem 2.4 as follows:

Remark 2.7. *Each conjugacy class $V \subset \mathbb{H}$ containing zeros of an $f \in \mathbb{H}[z]$ either contains exactly one left and exactly one right zero of f or $V \subset Z_{\ell}(f) \cap Z_{\mathbf{r}}(f)$.*

2.3. Minimal polynomials. Since the division algorithm (left and right) holds in $\mathbb{H}[z]$, any ideal (left or right) is principal. Given a set $\Lambda \subset \mathbb{H}$, the sets

$$\mathbb{I}_{\Lambda, \ell} := \{f \in \mathbb{H}[z] : Z_{\ell} \supset \Lambda\} \quad \text{and} \quad \mathbb{I}_{\Lambda, \mathbf{r}} := \{f \in \mathbb{H}[z] : Z_{\mathbf{r}} \supset \Lambda\} \quad (2.11)$$

are respectively, a right and a left ideal in $\mathbb{H}[z]$, which are non-trivial if and only if Λ is contained in a finite union of conjugacy classes. In the latter case, $\mathbb{I}_{\Lambda, \ell}$ and $\mathbb{I}_{\Lambda, \mathbf{r}}$ are generated by (unique) monic polynomials $P_{\Lambda, \ell}$ and $P_{\Lambda, \mathbf{r}}$ which will be called the *left* and the *right minimal polynomials* (abbreviated to **lmp** and **rmp**, respectively, in what follows) of Λ . They can be equivalently defined as unique monic polynomials of the lowest degree with the left (respectively, right) zero set equal Λ . Since $\mathbb{I}_{\emptyset, \ell} = \mathbb{I}_{\emptyset, \mathbf{r}} = \mathbb{H}[z]$, it makes sense to define the **lmp** and **rmp** of the empty set by letting $P_{\emptyset, \ell} = P_{\emptyset, \mathbf{r}} \equiv 1$. The next observation is a consequence of Lemma 2.6.

Remark 2.8. *Let $\Lambda \subset \mathbb{H}$ be contained in a finite union of conjugacy classes. If V is a conjugacy class disjoint with Λ and if $U \subset V$ contains at least two elements, then $P_{\Lambda \cup U, \ell}(z) = \mathcal{X}_V(z)P_{\Lambda, \ell}(z)$ and $P_{\Lambda \cup U, \mathbf{r}}(z) = \mathcal{X}_V(z)P_{\Lambda, \mathbf{r}}(z)$.*

Theorem 2.9. *Let $\Lambda \subset \mathbb{H}$ be arranged as $\Lambda = \{\alpha_1, \dots, \alpha_s\} \cup U_1 \cup \dots \cup U_k$, where U_1, \dots, U_k are subsets of disjoint conjugacy classes V_1, \dots, V_k respectively containing*

at least two elements each, and $\alpha_1, \dots, \alpha_s \in \mathbb{H} \setminus (V_1 \cup \dots \cup V_k)$ are non-equivalent quaternions. Then

$$P_{\Lambda, \ell}(z) = p_s(z) \cdot \prod_{j=1}^k \mathcal{X}_{V_j}(z) \quad \text{and} \quad P_{\Lambda, \mathbf{r}}(z) = q_s(z) \cdot \prod_{j=1}^k \mathcal{X}_{V_j}(z) \quad (2.12)$$

where p_s is the monic polynomial of degree s obtained from the recursion

$$p_0(z) \equiv 1, \quad p_{j+1}(z) = p_j(z) \left(z - p_j^{\mathbf{e}\ell}(\alpha_{j+1})^{-1} \alpha_{j+1} p_j^{\mathbf{e}\ell}(\alpha_{j+1}) \right), \quad (2.13)$$

and q_s is the monic polynomial of degree s obtained from the recursion

$$q_0(z) \equiv 1, \quad q_{j+1}(z) = (z - q_j^{\mathbf{e}\mathbf{r}}(\alpha_{j+1}) \alpha_{j+1} q_j^{\mathbf{e}\mathbf{r}}(\alpha_{j+1})^{-1}) \cdot q_j(z). \quad (2.14)$$

In particular, $\deg(P_{\Lambda, \ell}) = \deg(P_{\Lambda, \mathbf{r}}) = s + 2k$.

Proof. Based on the fact that all α_j 's belong to distinct conjugacy classes, the induction argument shows that for each $j = 1, \dots, s$, the polynomial $p_j(z)$ is the **imp** of the set $\{\alpha_1, \dots, \alpha_j\}$ and that $p_j^{\mathbf{e}\ell}(\alpha_{j+1}) \neq 0$ (so that the recursion formula (2.13) makes sense). Thus, p_s is the **imp** of the set $\{\alpha_1, \dots, \alpha_s\}$, and the repeated application of Remark 2.8 leads us to

$$P_{\Lambda, \ell}(z) = P_{\{\alpha_1, \dots, \alpha_s\}, \ell}(z) \cdot \prod_{j=1}^k \mathcal{X}_{V_j}(z) = p_s(z) \cdot \prod_{j=1}^k \mathcal{X}_{V_j}(z),$$

i.e., to the first formula in (2.12). The second formula follows in much the same way. The final statement of the theorem is an obvious consequence of (2.13), (2.14). \square

Recursion (2.13) was carried out under the assumption that all α_j 's are non-equivalent. It is worth noting that a slight modification of (2.13) applies to *any* finite set as follows: the **imp** of the set $\Lambda = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{H}$ equals the polynomial $p_n(z)$ obtained recursively by letting $p_0(z) \equiv 1$ and

$$p_{j+1}(z) = \begin{cases} p_j(z) \left(z - p_j^{\mathbf{e}\ell}(\alpha_{j+1})^{-1} \alpha_{j+1} p_j^{\mathbf{e}\ell}(\alpha_{j+1}) \right) & \text{if } p_j^{\mathbf{e}\ell}(\alpha_{j+1}) \neq 0, \\ p_j(z) & \text{if } p_j^{\mathbf{e}\ell}(\alpha_{j+1}) = 0. \end{cases} \quad (2.15)$$

We omit the straightforward inductive proof and the formulation of the right-sided version of (2.15). Instead, we include several final remarks.

It can be shown (again by induction) that $p_j^{\mathbf{e}\ell}(\alpha_{j+1}) = 0$ if only if at least two elements in $\{\alpha_1, \dots, \alpha_j\}$ are conjugates of α_{j+1} . Thus, the recursion (2.15) takes into account the two first elements from the same conjugacy class V and dismisses all further elements from V . In particular, the original recursion (2.13) applies to any finite set $\Lambda \subset \mathbb{H}$, none three elements of which are equivalent.

If V is a conjugacy class and if $V \cap \Lambda = \{\alpha_{i_1}, \alpha_{i_2}, \dots\}$, then $\mathcal{Z}_\ell(p_j) \cap V = \emptyset$ for $j < i_1$ and $\mathcal{Z}_\ell(p_j) \cap V = \{\alpha_{i_1}\}$ for $i_1 \leq j < i_2$. Since $\alpha_{i_2} \neq \alpha_{i_1}$, we have $p_{i_2-1}^{\mathbf{e}\ell}(\alpha_{i_2}) \neq 0$. The polynomial p_{i_2} defined by the top formula in (2.15) has two distinct left zeros $\alpha_{i_1}, \alpha_{i_2}$ in V and therefore, $V \subset \mathcal{Z}_\ell(p_j) \cap \mathcal{Z}_\mathbf{r}(p_j)$ for all $j \geq i_2$, by Lemma 2.6 and formulas (2.6), (2.7).

Recursion (2.15) produces the minimal polynomial $P_{\Lambda, \ell}$ as a product of linear factors. Although the outcome $P_{\Lambda, \ell}$ does not depend on the order in which the elements of Λ are arranged, different permutations of Λ produce via recursion (2.15) different factorizations of $P_{\Lambda, \ell}$. Factorization (2.12) comes up if Λ is arranged so that the two first appearances of the elements from each conjugacy class occur consecutively, that is, if α_j is not equivalent to $\alpha_1, \dots, \alpha_{j-1}$, then either $\alpha_j \sim \alpha_{j+1}$ or $\Lambda \setminus \{\alpha_j\}$ contains no element conjugate to α_j .

3. CONSISTENCY OF INTERPOLATION CONDITIONS AND SIMPLE CASES

In the complex setting, the Lagrange problem with distinct interpolation nodes is always consistent. In the quaternionic case, inconsistency may occur if the set $\Lambda \cup \Omega$ contains more than two points from the same conjugacy class; the one-sided version of this phenomenon was observed in [4] in a more general setting of slice regular functions.

Lemma 3.1. *For $f \in \mathbb{H}[z]$ and three distinct equivalent quaternions α, β, γ ,*

$$f^{e\ell}(\gamma) = (\gamma - \beta)(\alpha - \beta)^{-1}f^{e\ell}(\alpha) + (\alpha - \gamma)(\alpha - \beta)^{-1}f^{e\ell}(\beta), \quad (3.1)$$

$$\begin{aligned} f^{er}(\gamma) &= (\alpha - \beta)^{-1}f^{e\ell}(\alpha)\gamma - \beta(\alpha - \beta)^{-1}f^{e\ell}(\alpha) \\ &\quad + \alpha(\alpha - \beta)^{-1}f^{e\ell}(\beta) - (\alpha - \beta)^{-1}f^{e\ell}(\beta)\gamma, \end{aligned} \quad (3.2)$$

$$f^{er}(\gamma) = f^{er}(\alpha)(\alpha - \beta)^{-1}(\gamma - \beta) + f^{er}(\beta)(\alpha - \beta)^{-1}(\alpha - \gamma), \quad (3.3)$$

$$\begin{aligned} f^{e\ell}(\gamma) &= \gamma f^{er}(\alpha)(\alpha - \beta)^{-1} - f^{er}(\alpha)(\alpha - \beta)^{-1}\beta \\ &\quad + f^{er}(\beta)(\alpha - \beta)^{-1}\alpha - \gamma f^{er}(\beta)(\alpha - \beta)^{-1}. \end{aligned} \quad (3.4)$$

Proof. The polynomial $f(z) - f^{e\ell}(\alpha) - (z - \alpha)(\alpha - \beta)^{-1}(f^{e\ell}(\alpha) - f^{e\ell}(\beta))$ has left zeros at α and β and hence, by Lemma 2.6

$$f(z) = f^{e\ell}(\alpha) + (z - \alpha)(\alpha - \beta)^{-1}(f^{e\ell}(\alpha) - f^{e\ell}(\beta)) + \mathcal{X}_{[\alpha]}(z)g(z) \quad (3.5)$$

for some $g \in \mathbb{H}[z]$. Since $\gamma \in [\alpha]$, it holds that $(\mathcal{X}_{[\alpha]}g)^{e\ell}(\gamma) = (\mathcal{X}_{[\alpha]}g)^{er}(\gamma) = 0$ and then left and right evaluations of (3.5) at c give

$$\begin{aligned} f^{e\ell}(\gamma) &= f^{e\ell}(\alpha) + (\gamma - \alpha)(\alpha - \beta)^{-1}(f^{e\ell}(\alpha) - f^{e\ell}(\beta)), \\ f^{er}(\gamma) &= f^{e\ell}(\alpha) + (\alpha - \beta)^{-1}(f^{e\ell}(\alpha) - f^{e\ell}(\beta))\gamma \\ &\quad - \alpha(\alpha - \beta)^{-1}(f^{e\ell}(\alpha) - f^{e\ell}(\beta)), \end{aligned}$$

which are equivalent to (3.1) and (3.2), respectively. Relations (3.3) and (3.4) are established similarly, by applying Lemma 2.6 to the polynomial $f(z) - f^{er}(\alpha) - (f^{er}(\alpha) - f^{er}(\beta))(\alpha - \beta)^{-1}(z - \alpha)$. \square

Lemma 3.1 shows that left (or right) evaluations of $f \in \mathbb{H}[z]$ at any two points from the same conjugacy class uniquely determine left *and* right evaluations of f at any point in this conjugacy class. Thus, if the set $\Lambda \cup \Omega$ contains more than two points from the same conjugacy class, the corresponding target values must

satisfy certain conditions (outlined in Lemma 3.1) for the Lagrange problem to have a solution.

Let V be a conjugacy class such that $V \cap \Lambda = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\}$ contains at least two elements and let $V \cap \Omega = \{\beta_{j_1}, \dots, \beta_{j_s}\}$. For the assigned target values c_{i_ℓ} and d_{j_r} , we verify equalities (cf. (3.1) and (3.2))

$$\begin{aligned} c_{i_\ell} &= (\alpha_{i_\ell} - \alpha_{i_2})(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_1} + (\alpha_{i_1} - \alpha_{i_\ell})(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_2}, \\ d_{j_r} &= (\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_1}\beta_{j_r} - \alpha_{i_2}(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_1} \\ &\quad + \alpha_{i_1}(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_2} - (\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_2}\beta_{j_r} \end{aligned} \quad (3.6)$$

for $\ell = 3, \dots, k$ and $r = 1, \dots, s$. If at least one of them fails, the Lagrange problem (1.7), (1.8) does not have solutions, by Lemma 3.1. Otherwise, any polynomial $f \in \mathbb{H}[z]$ satisfying interpolation conditions (1.7) at α_{i_1} and α_{i_2} will satisfy left interpolation conditions at α_{i_ℓ} (for $\ell = 3, \dots, k$) and right interpolation conditions at β_{j_r} (for $r = 1, \dots, s$) automatically, again by Lemma 3.1. Hence, removing interpolation conditions at these points we get a reduced interpolation problem with the same solution set as the original one. Alternatively, if $V \cap \Omega$ contains at least two elements β_{j_1}, β_{j_2} , we may use relations (3.3) and (3.4) to check if other interpolation conditions on V are compatible with those two at β_{j_1} and β_{j_2} and, if this is the case, all other conditions can be removed without affecting the solution set of the problem. After completing consistency verifications in all conjugacy classes having more than two common elements with $\Lambda \cup \Omega$, we either conclude that the original problem is inconsistent or reduce it to a problem for which

(A) none three of the interpolation nodes are equivalent.

The latter assumption is all we need to handle one-sided interpolation problems.

3.1. One-sided interpolation. In case $\Omega = \emptyset$, all consistency equalities are of the form (3.6). Let us assume that all of them hold true and start with the left Lagrange problem satisfying the assumption (A).

Let $P_{\Lambda, \ell}$ be the **lmp** of Λ and let $P_{\Lambda_k, \ell}$ be the **lmp** of the set $\Lambda_k := \Lambda \setminus \{\alpha_k\}$ for $k = 1, \dots, n$. It follows from Theorem 2.9 that under assumption (A),

$$\deg(P_{\Lambda, \ell}) = n \quad \text{and} \quad \deg(P_{\Lambda_k, \ell}) = n - 1 \quad (k = 1, \dots, n). \quad (3.7)$$

Theorem 3.2. *Assume that none three elements in the set $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ are equivalent. All polynomials $f \in \mathbb{H}[z]$ satisfying left interpolation conditions (1.7) are given by the formula*

$$f = \tilde{f}_\ell + P_{\Lambda, \ell} h, \quad \tilde{f}_\ell(z) = \sum_{k=1}^n P_{\Lambda_k, \ell}(z) \cdot P_{\Lambda_k, \ell}^{e_\ell}(\alpha_k)^{-1} \cdot c_k, \quad h \in \mathbb{H}[z] \quad (3.8)$$

with free parameter $h \in \mathbb{H}[z]$. The left Lagrange polynomial \tilde{f}_ℓ is a unique solution to the problem (1.7) of degree less than n .

Proof. The polynomial $P_{\Lambda_k, \ell}$ left-vanishes on Λ_k , i.e., $P_{\Lambda_k, \ell}^{e_\ell}(\alpha_i) = 0$ for $i \neq k$. On the other hand, $P_{\Lambda_k, \ell}^{e_\ell}(\alpha_k) \neq 0$, since otherwise, the set $\tilde{\Lambda}_k$ contains at least two elements equivalent to α_k which contradicts the assumption (A). Now it is easily

verified that the polynomial \tilde{f}_ℓ defined as in (3.8) satisfies conditions (1.7). Due to (3.7), we have $\deg(\tilde{f}_\ell) < n$. A polynomial f satisfies conditions (1.7) if and only if $\mathcal{Z}_\ell(f - \tilde{f}_\ell) \supset \Lambda$, i.e., if and only if $f - \tilde{f}_\ell$ belongs to the right ideal $\mathbb{I}_{\Lambda, \ell}$ (2.11), which means that $f = \tilde{f}_\ell + P_{\Lambda, \ell} h$ for some $h \in \mathbb{H}[z]$. Finally, if $h \neq 0$, then $\deg(f) \geq \deg(P_{\Lambda, \ell}) = n > \deg(\tilde{f}_\ell)$. \square

The right-sided problem is handled in much the same way. Let $P_{\Omega, \mathbf{r}}^{er}$ be the **rmp** of $\Omega = \{\beta_1, \dots, \beta_m\}$ and let $P_{\Omega_k, \mathbf{r}}^{er}$ denote the **rmp** of the set $\Omega_j := \Omega \setminus \{\beta_j\}$ for $j = 1, \dots, m$. Under assumption **(A)**, we have

$$\deg(P_{\Omega, \mathbf{r}}) = m \quad \text{and} \quad \deg(P_{\Omega_j, \mathbf{r}}) = m - 1 \quad (j = 1, \dots, m). \quad (3.9)$$

Theorem 3.3. *Assume that none three elements in the set $\Omega = \{\beta_1, \dots, \beta_m\}$ are equivalent. Then all $f \in \mathbb{H}[z]$ satisfying right conditions (1.8) are given by the formula*

$$f = \tilde{f}_{\mathbf{r}} + h P_{\Lambda, \mathbf{r}}, \quad \tilde{f}_{\mathbf{r}}(z) = \sum_{k=1}^m d_k \cdot P_{\Omega_k, \mathbf{r}}^{er}(\beta_k)^{-1} \cdot P_{\Omega_k, \mathbf{r}}(z), \quad h \in \mathbb{H}[z] \quad (3.10)$$

with free parameter $h \in \mathbb{H}[z]$. The right Lagrange polynomial $\tilde{f}_{\mathbf{r}}$ is a unique solution to the problem (1.8) of degree less than m .

4. BACKWARD SHIFT OPERATORS AND SYLVESTER EQUATIONS

With any $\alpha \in \mathbb{H}$, we can associate linear operators L_α and R_α (left and right backward shifts) acting on $\mathbb{H}[z]$ (which is now considered as a linear vector space over \mathbb{H}):

$$L_\alpha : f(z) = \sum_{k=0}^n z^k f_k \mapsto \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-k-1} \alpha^j f_{k+j+1} \right) z^k, \quad (4.1)$$

$$R_\alpha : f(z) = \sum_{k=0}^n z^k f_k \mapsto \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n-k-1} f_{k+j+1} \alpha^j \right) z^k. \quad (4.2)$$

The terminology is partly justified by the fact that $L_\alpha f$ and $R_\alpha f$ are the unique polynomials such that

$$f(z) = f^{e\ell}(\alpha) + (z - \alpha) \cdot (L_\alpha f)(z) = f^{er}(\alpha) + (R_\alpha f)(z) \cdot (z - \alpha). \quad (4.3)$$

It follows directly from (4.1) that for any $\alpha, \beta \in \mathbb{H}$,

$$(L_\alpha f)^{er}(\beta) = (R_\beta f)^{e\ell}(\alpha) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} \alpha^k f_{k+j+1} \beta^{k-j}. \quad (4.4)$$

Remark 4.1. *For $f \in \mathbb{H}[z]$ and $\alpha, \beta \in \mathbb{H}$,*

$$\alpha \cdot (L_\alpha f)^{er}(\beta) - (L_\alpha f)^{er}(\beta) \cdot \beta = f^{e\ell}(\alpha) - f^{er}(\beta). \quad (4.5)$$

Indeed, the right evaluation at $z = \beta$ applied to the first representation in (4.3) gives $f^{er}(\beta) = f^{el}(\alpha) + (L_\alpha f)^{er}(\beta) \cdot \beta - \alpha \cdot (L_\alpha f)^{er}(\beta)$ which is equivalent to (4.5). Our next objective is to determine to what extent the value of $(L_\alpha f)^{er}(\beta)$ can be recovered from the equality (4.5). The next result is known.

Lemma 4.2. *Given two non-equivalent $\alpha, \beta \in \mathbb{H}$, the Sylvester equation*

$$\alpha q - q\beta = \Delta \quad (4.6)$$

has a unique solution $q = (\bar{\alpha}\Delta - \Delta\beta)\mathcal{X}_{[\alpha]}(\beta)^{-1}$ for any $\Delta \in \mathbb{H}$. Consequently, if $f \in \mathbb{H}[z]$ and $\alpha \not\sim \beta$, then

$$(L_\alpha f)^{er}(\beta) = (\bar{\alpha}(f^{el}(\alpha) - f^{er}(\beta)) - (f^{el}(\alpha) - f^{er}(\beta))\beta)\mathcal{X}_{[\alpha]}(\beta)^{-1}. \quad (4.7)$$

Proof. Multiplying both parts of (4.6) by $\bar{\alpha}$ on the left and by β on the right gives

$$|\alpha|^2 q - \bar{\alpha}q\beta = \bar{\alpha}\Delta \quad \text{and} \quad \alpha q\beta - q\beta^2 = \Delta\beta,$$

respectively. Subtracting the second equation from the first and commuting real coefficients we get

$$\begin{aligned} \bar{\alpha}\Delta - \Delta\beta &= |\alpha|^2 q - (\alpha + \bar{\alpha})q\beta + q\beta^2 \\ &= q(|\alpha|^2 - 2\beta\text{Re}(\alpha) + \beta^2) = q\mathcal{X}_{[\alpha]}(\beta), \end{aligned} \quad (4.8)$$

and the desired formula for q follows since $\alpha \not\sim \beta$ and therefore, $\mathcal{X}_{[\alpha]}(\beta) \neq 0$. Applying this formula to the equation (4.5) (i.e., for $\Delta = f^{el}(\alpha) - f^{er}(\beta)$), we get (4.7). \square

The case where α and β are equivalent is more interesting. Let us denote by \mathbb{S} the unit sphere of purely imaginary quaternions. Any $I \in \mathbb{S}$ is such that $I^2 = -1$. Recall that if α and β are two equivalent quaternions, then due to characterization (2.1), they can be written in the form

$$\alpha = x + yI, \quad \beta = x + y\tilde{I} \quad (x \in \mathbb{R}, y > 0, I, \tilde{I} \in \mathbb{S}). \quad (4.9)$$

Since \mathbb{H} is a (four-dimensional) vector space over \mathbb{R} , we may define orthogonal complements with respect to the usual euclidean metric in \mathbb{R}^4 . For α and β as in (4.9), we define the plane (the two-dimensional subspace of $\mathbb{H} \cong \mathbb{R}^4$) $\Pi_{\alpha,\beta}$ via the formula

$$\Pi_{\alpha,\beta} = \begin{cases} \text{span}\{1, I\} = \{u + vI : u, v \in \mathbb{R}\}, & \text{if } \beta = \alpha, \\ (\text{span}\{1, I\})^\perp, & \text{if } \beta = \bar{\alpha}, \\ \text{span}\{I + \tilde{I}, 1 - I\tilde{I}\}, & \text{if } \beta \neq \alpha, \bar{\alpha}. \end{cases} \quad (4.10)$$

Since $\bar{\alpha} = x - yI$, it follows that $\Pi_{\bar{\alpha},\bar{\alpha}} = \Pi_{\alpha,\alpha}$, $\Pi_{\bar{\alpha},\alpha} = \Pi_{\alpha,\bar{\alpha}}$ and

$$\Pi_{\bar{\alpha},\beta} = \text{span}\{I - \tilde{I}, 1 + I\tilde{I}\} \quad \text{if } \beta \neq \alpha, \bar{\alpha}. \quad (4.11)$$

Lemma 4.3. *Let $\alpha \sim \beta$ be of the form (4.9). Then the Sylvester equation (4.3) has a solution if and only if $\Delta \in \Pi_{\bar{\alpha},\beta}$ or equivalently, if and only if*

$$\bar{\alpha}\Delta = \Delta\beta. \quad (4.12)$$

If this is the case, the solution set for the equation (4.6) is the affine plane

$$(2\text{Im}(\alpha))^{-1}\Delta + \Pi_{\alpha,\beta} = -\Delta(2\text{Im}(\beta))^{-1} + \Pi_{\alpha,\beta}.$$

Proof. If $\alpha \sim \beta$, then $\mathcal{X}_{[\alpha]}(\beta) = 0$, and calculation (4.8) shows that we necessarily have (4.12). Substituting (4.9) into (4.12) gives $-yI\Delta = y\Delta\tilde{I}$, which is equivalent (since $y \neq 0$ and $I^2 = -1$) to $\Delta = I\Delta\tilde{I}$. Therefore,

$$\alpha I\Delta - I\Delta\beta = yI^2\Delta - yI\Delta\tilde{I} = -2y\Delta$$

and hence, the element $q_0 = (2\text{Im}(\alpha))^{-1}\Delta = -\Delta(2\text{Im}(\beta))^{-1}$ is a particular solution of the equation (4.6). It remains to show that the solution set of the homogeneous Sylvester equation

$$\alpha p - p\beta = 0 \tag{4.13}$$

coincides with the plane $\Pi_{\alpha,\beta}$ defined in (4.10). To this end, we first observe that since \mathbb{R} is the center of \mathbb{H} , it follows that the solution set $\Omega_{\alpha,\beta}$ of the equation (4.13) is a (real) subspace of $\mathbb{H} \cong \mathbb{R}^4$. For a given $I \in \mathbb{S}$ and any $J \in \mathbb{S}$ which is orthogonal to I (as a vector in $\mathbb{H} \cong \mathbb{R}^4$), the elements $\{1, I, J, IJ\}$ form an orthonormal basis in \mathbb{H} and therefore, any element $p \in \mathbb{H}$ admits a unique representation

$$p = x_0 + x_1I + x_2J + x_3IJ \quad (x_0, x_1, x_2, x_3 \in \mathbb{R}) \tag{4.14}$$

similar to (1.4). Observe that since $I, J \in \mathbb{S}$ and $I \perp J$, we have $IJ = -JI$. We then use the latter equality along with representations (4.9) and (4.14) to compute

$$\alpha p - p\alpha = 2y(x_2IJ - x_3J) \quad \text{and} \quad \alpha p - p\bar{\alpha} = 2y(x_0I - x_1).$$

Thus, $\alpha p = p\alpha$ if and only if $x_2 = x_3 = 0$ and hence, $p \in \text{span}\{1, I\}$, and on the other hand, $\alpha p = p\bar{\alpha}$ if and only if $x_0 = x_1 = 0$ and hence, $p \in \text{span}\{J, IJ\} = (\text{span}\{1, I\})^\perp$, which proves that $\Omega_{\alpha,\beta}$ indeed is equal to the plane $\Pi_{\alpha,\beta}$ for the cases where $\beta = \alpha$ or $\beta = \bar{\alpha}$.

For the remaining case, we will argue as follows. Since $\beta \neq \alpha, \bar{\alpha}$, representations (4.9) hold with $\tilde{I} \neq \pm I$. Letting $p_1 = I + \tilde{I}$ and $p_2 = 1 - I\tilde{I}$ we see that

$$\alpha p_1 - p_1\beta = y(I^2 + I\tilde{I} - I\tilde{I} - \tilde{I}^2) = 0,$$

$$\alpha p_2 - p_2\beta = y(I - I^2\tilde{I} - \tilde{I} + I\tilde{I}^2) = 0,$$

since $I^2 = \tilde{I}^2 = -1$. Thus, p_1 and p_2 are linearly independent (over \mathbb{R}) solutions of the equation (4.13) and therefore $\Omega_{\alpha,\beta} \supset \Pi_{\alpha,\beta}$, so that $\dim \Omega_{\alpha,\beta} \geq \dim \Pi_{\alpha,\beta} = 2$.

Similarly, one can verify that $p_3 = I - \tilde{I}$ and $p_4 = 1 + I\tilde{I}$ are two linear independent solutions to the Sylvester equation $\bar{\alpha}p = p\beta$, the solution set $\Omega_{\bar{\alpha},\beta}$ of which contains the plane $\Pi_{\bar{\alpha},\beta}$ (see (4.11)) and therefore, is a subspace of \mathbb{H} of dimension of at least two. Observe that since $\alpha \neq \bar{\alpha}$, equalities $\alpha p = p\beta = \bar{\alpha}p$ imply $p = 0$. Therefore $\Omega_{\alpha,\beta} \cap \Omega_{\bar{\alpha},\beta} = \{0\}$, and consequently, $\dim \Omega_{\alpha,\beta} = \dim \Omega_{\bar{\alpha},\beta} = 2$. Therefore, $\Omega_{\alpha,\beta} = \Pi_{\alpha,\beta}$ and $\Omega_{\bar{\alpha},\beta} = \Pi_{\bar{\alpha},\beta}$. In particular, Δ is subject to condition (4.12) if and only if it belongs to $\Pi_{\bar{\alpha},\beta}$. \square

Corollary 4.4. *For $f \in \mathbb{H}[z]$ and $\alpha \sim \beta \in \mathbb{H}$,*

$$\overline{\alpha}(f^{e\ell}(\alpha) - f^{er}(\beta)) = (f^{e\ell}(\alpha) - f^{er}(\beta))\beta.$$

Proof. Equality (4.5) tells us that for $\Delta := f^{e\ell}(\alpha) - f^{er}(\beta)$, the Sylvester equation (4.6) has a solution. By Lemma (4.3), equality (4.12) holds, which is the same as (4.9), due to the present choice of Δ . \square

We may now present necessary and sufficient conditions for the two-sided Lagrange problem to have a solution.

Theorem 4.5. *Assume that none three elements of the set $\Lambda \cup \Omega$ are equivalent. There is a polynomial $f \in \mathbb{H}[z]$ satisfying conditions (1.7), (1.8) if and only if $\overline{\alpha}_i(c_i - d_j) = (c_i - d_j)\beta_j$ for each pair $(\alpha_i, \beta_j) \in \Lambda \times \Omega$ of equivalent nodes.*

The “only if” part follows from Corollary 4.4. The sufficiency part will be confirmed in the next section.

5. TWO-SIDED PROBLEM

We still assume that none three of interpolation nodes are equivalent. To be more specific, we assume that there are k equivalent pairs in $\Lambda \times \Omega$ (the case $k = 0$ is not excluded), and we rearrange the sets Λ and Ω (1.6) so that these equivalent pairs are (α_i, β_i) for $i = 1, \dots, k$. In other words,

$$\alpha_i \sim \beta_i \quad (1 \leq i \leq k); \quad [\alpha_i] \cap \Omega = \emptyset \quad (k < i \leq n); \quad [\beta_j] \cap \Lambda = \emptyset \quad (k < j \leq m). \quad (5.1)$$

We also assume that the necessary conditions from Theorem 4.5 hold:

$$\overline{\alpha}_i(c_i - d_i) = (c_i - d_i)\beta_i \quad (i = 1, \dots, k). \quad (5.2)$$

One may try to handle the two-sided problem in a standard way by combining the explicit formula for the particular low degree solution and the parametrization of the solution set for the homogeneous problem. Both ingredients are not as simple as in one-sided cases.

5.1. Homogeneous problems. The homogeneous counterpart of the two-sided Lagrange problem (1.7), (1.8) consists of finding all $f \in \mathbb{H}[z]$ such that

$$f^{e\ell}(\alpha_i) = 0 \quad (1 \leq i \leq n); \quad f^{er}(\beta_j) = 0 \quad (1 \leq j \leq m). \quad (5.3)$$

The set $\mathbb{I}_{\Lambda, \ell}$ of all polynomials $f \in \mathbb{H}[z]$ satisfying left conditions in (5.3) is the right ideal generated by the **imp** $P_{\Lambda, \ell}$ of $\Lambda = \{\alpha_1, \dots, \alpha_n\}$; in fact, $\mathbb{I}_{\Lambda, \ell}$ is the finite intersection of maximal right ideals in $\mathbb{H}[z]$:

$$\mathbb{I}_{\Lambda, \ell} = \bigcap_{i=1}^n \{f \in \mathbb{H}[z] : f(\alpha_i) = 0\} = P_{\Lambda, \ell} \cdot \mathbb{H}[z]. \quad (5.4)$$

Analogously, the set $\mathbb{I}_{\Omega, \mathbf{r}}$ of all polynomials satisfying right conditions in (5.3) is the left ideal (in fact, the finite intersection of maximal left ideals) in $\mathbb{H}[z]$ generated by the **rmf** $P_{\Omega, \mathbf{r}}$ of the set $\Omega = \{\beta_1, \dots, \beta_m\}$:

$$\mathbb{I}_{\Omega, \mathbf{r}} = \bigcap_{j=1}^m \{f \in \mathbb{H}[z] : f(\beta_j) = 0\} = \mathbb{H}[z] \cdot P_{\Omega, \mathbf{r}}. \quad (5.5)$$

Hence, $\mathbb{I}_{\Lambda, \ell} \cap \mathbb{I}_{\Omega, \mathbf{r}}$ is the solution set to the problem (5.3) and the next question is to describe this intersection analytically. The next result shows that in case $k = 0$ in (5.1), $\mathbb{I}_{\Lambda, \ell} \cap \mathbb{I}_{\Omega, \mathbf{r}} = P_{\Lambda, \ell} \cdot \mathbb{H}[z] \cdot P_{\Omega, \mathbf{r}}$.

Theorem 5.1. *A polynomial $f \in \mathbb{H}[z]$ satisfies (5.3) and additional conditions*

$$(L_{\alpha_i} f)^{er}(\beta_i) = 0 \quad \text{for } i = 1, \dots, k \quad (5.6)$$

if and only if it belongs to $P_{\Lambda, \ell} \cdot \mathbb{H}[z] \cdot P_{\Omega, \mathbf{r}}$.

Proof. For any $h \in \mathbb{H}[z]$, the polynomial $f = P_{\Lambda, \ell} \cdot h \cdot P_{\Omega, \mathbf{r}}$ clearly satisfies conditions (5.3). Since $P_{\Lambda, \ell}^{el}(\alpha_i) = 0$, we have

$$L_{\alpha_i} f = L_{\alpha_i}(P_{\Lambda, \ell} \cdot h \cdot P_{\Omega, \mathbf{r}}) = (L_{\alpha_i} P_{\Lambda, \ell}) \cdot h \cdot P_{\Omega, \mathbf{r}} \quad (5.7)$$

and since $P_{\Omega, \mathbf{r}}^{er}(\beta_i) = 0$, the right evaluation at $z = \beta_i$ applied to both sides of (5.7) implies (5.6).

Conversely, let $f \in \mathbb{H}[z]$ satisfy conditions (5.3), (5.6). Due to the left conditions in (5.3), f is in $\mathbb{I}_{\Lambda, \ell}$ and thus, it is of the form $f = P_{\Lambda, \ell} \cdot g$ for some $g \in \mathbb{H}[z]$. To complete the proof, it suffices to show that $g \in \mathbb{I}_{\Omega, \mathbf{r}}$.

If $j > k$, (i.e., if $[\beta_j] \cap \Lambda = \emptyset$; see (5.1)), then $P_{\Lambda, \ell}$ does not have zeros (either left or right, by Remark (2.7)) in the conjugacy class $[\beta_j]$. Then the equality

$$f^{er}(\beta_j) = (P_{\Lambda, \ell} \cdot g)^{er}(\beta_j) = 0$$

implies $g^{er}(\beta_j) = 0$ since otherwise, the polynomial $P_{\Lambda, \ell}$ had a right zero at

$$g^{er}(\beta_j) \beta_j g^{er}(\beta_j)^{-1} \in [\beta_j] \quad (5.8)$$

which is a contradiction. Thus, $g^{er}(\beta_j) = 0$ for all $j = k+1, \dots, m$.

Observe that $\mathcal{Z}_\ell(L_{\alpha_j} P_{\Lambda, \ell}) \cup \mathcal{Z}_\mathbf{r}(L_{\alpha_j} P_{\Lambda, \ell}) \subset \bigcup_{i \neq j} [\alpha_i]$. Hence, if $j \leq k$ (i.e., $\alpha_j \sim \beta_j$ and $\alpha_i \not\sim \alpha_j$ whenever $i \neq j$), then we have

$$(\mathcal{Z}_\ell(L_{\alpha_j} P_{\Lambda, \ell}) \cup \mathcal{Z}_\mathbf{r}(L_{\alpha_j} P_{\Lambda, \ell})) \cap [\alpha_j] = \emptyset. \quad (5.9)$$

Then the equality

$$(L_{\alpha_j} f)^{er}(\beta_j) = (L_{\alpha_j}(P_{\Lambda, \ell} \cdot g))^{er}(\beta_j) = ((L_{\alpha_j} P_{\Lambda, \ell}) \cdot g)^{er}(\beta_j) = 0$$

implies $g^{er}(\beta_j) = 0$, since otherwise, $L_{\alpha_j} P_{\Lambda, \ell}$ has right zero at the point (5.8) which contradicts (5.9). We thus have $g^{er}(\beta_j) = 0$ for all $j = 1, \dots, m$ and hence, $g \in \mathbb{I}_{\Omega, \mathbf{r}}$ which completes the proof. \square

If $k > 0$ in (5.1), then the set $P_{\Lambda, \ell} \cdot \mathbb{H}[z] \cdot P_{\Omega, \mathbf{r}}$ is properly included in $\mathbb{I}_{\Lambda, \ell} \cap \mathbb{I}_{\Omega, \mathbf{r}}$. The analytic description of $\mathbb{I}_{\Lambda, \ell} \cap \mathbb{I}_{\Omega, \mathbf{r}}$ is given below. Recall that $P_{\Lambda_i, \ell}$ is the **imp** of the set $\Lambda_i = \Lambda \setminus \{\alpha_i\}$ and $P_{\Omega_i, \mathbf{r}}$ is the **rmf** of $\Omega_i = \Omega \setminus \{\beta_i\}$.

Theorem 5.2. *Under assumptions (5.1), let*

$$\tilde{\alpha}_i = P_{\Lambda_i, \ell}^{e_\ell}(\alpha_i)^{-1} \cdot \alpha_i \cdot P_{\Lambda_i, \ell}^{e_\ell}(\alpha_i), \quad \tilde{\beta}_i = P_{\Omega_i, \mathbf{r}}^{e_r}(\beta_i) \cdot \beta_i \cdot P_{\Omega_i, \mathbf{r}}^{e_r}(\beta_i)^{-1}, \quad (5.10)$$

so that $\tilde{\alpha}_i \sim \alpha_i \sim \beta_i \sim \tilde{\beta}_i$. Let $\Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}$ be the plane defined via formula (4.10) for $i = 1, \dots, k$. Then f belongs to $\mathbb{I}_{\Lambda, \ell} \cap \mathbb{I}_{\Omega, \mathbf{r}}$ if and only if it is of the form

$$f(z) = \sum_{i=1}^k P_{\Lambda, \ell}(z) \cdot \mu_i \cdot P_{\Omega_i, \mathbf{r}}(z) + P_{\Lambda, \ell}(z) \cdot h(z) \cdot P_{\Omega, \mathbf{r}}(z) \quad (5.11)$$

for some $h \in \mathbb{H}[z]$ and $\mu_i \in \Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}$.

The proof will be given in Section 6 as a consequence of Theorem 5.9. Although the result of Theorem 5.2 will not be used in our further analysis, it is of some independent interest as we now explain.

5.2. Quasi-ideals and principal bi-ideals in $\mathbb{H}[z]$. The notions of quasi-ideals and bi-ideals in associative rings were introduced in [11] and [6], respectively. We recall these notions in the present context of $R = \mathbb{H}[z]$ (a ring with identity, in which any ideal is principal).

A subset of $\mathbb{H}[z]$ is called a *bi-ideal* if it is a left ideal of some right ideal in $\mathbb{H}[z]$ or equivalently, it is a right ideal of a left ideal in $\mathbb{H}[z]$. Although all ideals in $\mathbb{H}[z]$ are principal, a left ideal of a right ideal of $\mathbb{H}[z]$ is not principal, in general. Let us say that Q is a *principal bi-ideal* (this notion is not common) if it is a principal left ideal of some right ideal in $\mathbb{H}[z]$. According to this definition, each principal bi-ideal in $\mathbb{H}[z]$ is of the form $Q = p \cdot \mathbb{H}[z] \cdot q$ for some fixed $p, q \in \mathbb{H}[z]$ and hence, a principal bi-ideal can be equivalently defined as a principal right ideal of some left ideal in $\mathbb{H}[z]$. Theorem 5.1 tells us that the solution set of the homogeneous interpolation problem (5.3), (5.6) is a principal bi-ideal.

Recall that a subset of a ring R with identity is called a *quasi-ideal* if it is the intersection of a left ideal of R with a right ideal of R . Theorem 5.2 provides an analytic description of a quasi-ideal in $\mathbb{H}[z]$ which is the intersection of finitely many maximal left ideals and finitely many maximal right ideals in $\mathbb{H}[z]$.

Quasi-ideals and principal bi-ideals are special instances of bi-ideals. In general, these two notions are distinct. To demonstrate this, let

$$\mathbb{I}_\ell = (z - \mathbf{i}) \cdot \mathbb{H}[z], \quad \mathbb{I}_\mathbf{r} = \mathbb{H}[z] \cdot (z - \mathbf{j}), \quad Q = (z - \mathbf{i}) \cdot \mathbb{I}_\mathbf{r} = \mathbb{I}_\ell \cdot (z - \mathbf{j}),$$

and let us show that

- (1) the principal bi-ideal Q is not a quasi-ideal and that
- (2) the quasi-ideal $\mathbb{I}_\ell \cap \mathbb{I}_\mathbf{r}$ is not a principal bi-ideal.

We have $Q \subset \mathbb{I}_\ell \cap \mathbb{I}_\mathbf{r}$, and the inclusion is proper since the function

$$(z - \mathbf{i})(\mathbf{i} + \mathbf{j}) = (\mathbf{i} + \mathbf{j})(z - \mathbf{j}) \quad \text{belongs to} \quad \mathbb{I}_\ell \cap \mathbb{I}_\mathbf{r}, \quad (5.12)$$

but not to Q . Let us assume that there is a proper right ideal $\tilde{\mathbb{I}}_\ell \subset \mathbb{I}_\ell$ containing Q . Then $\tilde{\mathbb{I}}_\ell$ is generated by a right multiple of $(z - \mathbf{i})$ and hence $L_\mathbf{i}Q = \mathbb{I}_\mathbf{r} \subset (z - \alpha) \cdot \mathbb{H}[z]$ for some $\alpha \in \mathbb{H}$. Since $h(z) = z - \mathbf{j}$ belongs to $\mathbb{I}_\mathbf{r}$, we necessarily have $\alpha = \mathbf{j}$. Since

$g(z) = \mathbf{i}(z - \mathbf{j}) = (z + \mathbf{j})\mathbf{i}$ belongs to \mathbb{I}_r , we also have $\alpha = -\mathbf{j}$ which is a contradiction. Since the intersection of right ideals is a right ideal, it follows that any right ideal in $\mathbb{H}[z]$ containing Q also contains \mathbb{I}_ℓ . Similarly, any left ideal in $\mathbb{H}[z]$ containing Q also contains \mathbb{I}_r . Therefore, $\mathbb{I}_\ell \cap \mathbb{I}_r$ is the minimal quasi-ideal containing Q , and since $Q \neq \mathbb{I}_\ell \cap \mathbb{I}_r$, it follows that Q is not a quasi-ideal.

To show part (2), we first observe that $\mathbb{I}_\ell \cap \mathbb{I}_r$ is not a left ideal in $\mathbb{H}[z]$. Indeed, if $\mathbb{I}_\ell \cap \mathbb{I}_r$ were a left ideal, it would have been a proper ideal of \mathbb{I}_r generated therefore by a polynomial of degree at least two and not containing therefore, polynomials of degree one. The latter contradicts to (5.12). Similarly, $\mathbb{I}_\ell \cap \mathbb{I}_r$ is not a right ideal either. If $\mathbb{I}_\ell \cap \mathbb{I}_r$ is a principal bi-ideal, it admits a representation $\mathbb{I}_\ell \cap \mathbb{I}_r = p \cdot \mathbb{H}[z] \cdot q$ for some polynomials p, q of degree at least one. Then it again cannot contain polynomials of degree one which contradicts to (5.12). Therefore, $\mathbb{I}_\ell \cap \mathbb{I}_r$ is not a principal bi-ideal.

5.3. Elementary cases. In formula (1.2), the complex Lagrange interpolation polynomial \tilde{f} is constructed as the sum of polynomials $\tilde{f}_j(z) = \frac{c_j p_j(z)}{p_j(z_j)}$ satisfying interpolation conditions $\tilde{f}_j(z_j) = c_j$ and $\tilde{f}_j(z_i) = 0$ for $i \neq j$. The formulas (3.8) for the left Lagrange polynomial \tilde{f}_ℓ and (3.10) for the right Lagrange polynomial \tilde{f}_r followed the same strategy: to construct “elementary” polynomials satisfying one non-homogeneous interpolation condition from (1.7) (respectively, from (1.8)) and having left (respectively, right) zeros at all other interpolation nodes, and then to construct \tilde{f}_ℓ (respectively, \tilde{f}_r) as the sum of these polynomials. In this section we will adapt this approach to the two-sided problem. We start with a technical result.

Lemma 5.3. *Given $P \in \mathbb{H}[z]$, given $\beta \notin \mathcal{Z}(P^\sharp P)$ and non-zero $d, \delta \in \mathbb{H}$,*

$$d = P^{e_r}(\delta\beta\delta^{-1}) \cdot \delta \iff \delta = P^{\sharp e_r}(d\beta d^{-1}) \cdot d \cdot (P^\sharp P)(\beta)^{-1}; \quad (5.13)$$

$$d = \delta \cdot P^{e_\ell}(\delta^{-1}\beta\delta) \iff \delta = (P^\sharp P)(\beta)^{-1} \cdot d \cdot P^{\sharp e_\ell}(d^{-1}\beta d). \quad (5.14)$$

Proof. If $d = P^{e_r}(\delta\beta\delta^{-1}) \cdot \delta$, then

$$\begin{aligned} P^{\sharp e_r}(d\beta d^{-1}) \cdot d &= P^{\sharp e_r}(P^{e_r}(\delta\beta\delta^{-1})\delta\beta\delta^{-1}P^{e_r}(\delta\beta\delta^{-1})^{-1})P^{e_r}(\delta\beta\delta^{-1})\delta \\ &= (P^\sharp P)^{e_r}(\delta\beta\delta^{-1}) \cdot \delta = \delta \cdot (P^\sharp P)(\beta), \end{aligned}$$

where the second equality follows from formula (2.7), and the third equality follows since the polynomial $P^\sharp P$ is real. Since $(P^\sharp P)(\beta) \neq 0$, the latter formula implies the formula for δ in (5.13) which completes the proof of implication \Rightarrow in (5.13). To prove the reverse implication, we observe that the right equality in (5.13) is equivalent to

$$\delta \cdot (P^\sharp P)(\beta) = P^{\sharp e_r}(d\beta d^{-1}) \cdot d.$$

We then apply the implication \Rightarrow (just proven) to the latter equality, i.e., to P^\sharp , $\delta \cdot (P^\sharp P)(\beta)$ and d instead of P , d and δ , respectively:

$$\begin{aligned} d &= P^{er} \left(\delta \cdot (P^\sharp P)(\beta) \cdot \beta \cdot (P^\sharp P)(\beta)^{-1} \cdot \delta^{-1} \right) \cdot \delta \cdot (P^\sharp P)(\beta) \cdot (P^\sharp P)(\beta)^{-1} \\ &= P^{er} (\delta \beta \delta^{-1}) \cdot \delta, \end{aligned}$$

where the second equality holds since β and $(P^\sharp P)(\beta)$ commute. This completes the proof of the equivalence (5.13). If we apply this equivalence to P^\sharp and quaternionic conjugates of β, d, δ , we get

$$\bar{d} = P^{\sharp er} \left(\overline{\delta^{-1} \beta \delta} \right) \cdot \bar{\delta} \iff \bar{\delta} = P^{er} \left(\overline{d^{-1} \beta d} \right) \cdot \bar{d} \cdot \left[(P P^\sharp)(\bar{\beta}) \right]^{-1}.$$

Taking quaternionic conjugates in both equalities and making use of equality $f^{e\ell}(\alpha) = \overline{f^{\sharp er}(\bar{\alpha})}$ (holding for all $\alpha \in \mathbb{H}$, due to (2.2)), we arrive at (5.14). \square

The next three lemmas present “elementary” polynomials which then will be used to construct a family of low-degree solutions to the problem (1.7), (1.8).

Lemma 5.4. *Under assumptions (5.1), let $s \in \{k+1, \dots, m\}$ and let*

$$\gamma_s = \begin{cases} P_{\Lambda, \ell}^{\sharp er}(d_s \beta_s d_s^{-1}) \cdot d_s \cdot (P_{\Lambda, \ell}^\sharp P_{\Lambda, \ell})(\beta_s)^{-1} \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s)^{-1}, & \text{if } d_s \neq 0, \\ 0, & \text{if } d_s = 0. \end{cases} \quad (5.15)$$

Then

$$\tilde{f}_{\mathbf{r}, s}(z) = P_{\Lambda, \ell}(z) \cdot \gamma_s \cdot P_{\Omega_s, \mathbf{r}}(z) \quad (5.16)$$

is a unique polynomial of degree less than $n + m$ satisfying conditions

$$f^{er}(\beta_s) = d_s, \quad f^{er}(\beta_i) = 0 \quad (i \neq j), \quad f^{e\ell}(\alpha_i) = 0 \quad (i = 1, \dots, n), \quad (5.17)$$

$$(L_{\alpha_i} f)^{er}(\beta_i) = 0 \quad \text{for } i = 1, \dots, k. \quad (5.18)$$

Proof. If $d_s = 0$, we have the homogeneous interpolation problem (5.3), (5.6), and the statement follows from Theorem 5.1. Assume that $d_s \neq 0$. Since none three elements in Ω are equivalent, the set $\mathcal{Z}_{\mathbf{r}}(P_{\Omega_s, \mathbf{r}}) = \Omega_s = \Omega \setminus \{\beta_s\}$ contains at most one conjugate of β_s ; therefore, $P_{\Omega_s, \mathbf{r}}(\beta_s) \neq 0$. Since $s > k$, we have by assumption (5.1), $\beta_s \notin \bigcup_{i=1}^n [\alpha_i] = \mathcal{Z}(P_{\Lambda, \ell}^\sharp P_{\Lambda, \ell})$ and therefore, $(P_{\Lambda, \ell}^\sharp P_{\Lambda, \ell})(\beta_s) \neq 0$. Hence, the formula (5.15) makes sense.

To show that $\tilde{f}_{\mathbf{r}, s}$ defined as in (5.16) satisfies the first condition in (5.17), let write (5.15) (recall that $d_s \neq 0$) equivalently as

$$\gamma_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s) = P_{\Lambda, \ell}^{\sharp er}(d_s \beta_s d_s^{-1}) \cdot d_s \cdot (P_{\Lambda, \ell}^\sharp P_{\Lambda, \ell})(\beta_s)^{-1}. \quad (5.19)$$

By implication \Leftarrow in (5.13) (with $\delta = \gamma_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s)$), we then have from (5.19)

$$d_s = P_{\Lambda, \ell}^{er} \left(\gamma_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s) \cdot \beta_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s)^{-1} \cdot \gamma_s^{-1} \right) \cdot \gamma_j \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s).$$

On the other hand, applying formula (2.7) to the product in (5.16) gives

$$\tilde{f}_{\mathbf{r}, s}^{er}(\beta_s) = P_{\Lambda, \ell}^{er} \left(\gamma_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s) \cdot \beta_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s)^{-1} \cdot \gamma_s^{-1} \right) \cdot \gamma_s \cdot P_{\Omega_s, \mathbf{r}}^{er}(\beta_s).$$

The two last equalities imply $\tilde{f}_{\mathbf{r},s}^{e_r}(\beta_s) = d_s$ so that $\tilde{f}_{\mathbf{r},s}$ indeed satisfies the first condition in (5.16). Other conditions in (5.16) are met since $P_{\Lambda,\ell}^{e_\ell}(\alpha_i) = 0$ (for all $\alpha_i \in \Lambda$) and $P_{\Omega_s,\mathbf{r}}^{e_r}(\beta_i) = 0$ (for all $\beta_i \in \Omega \setminus \{\beta_s\}$) by definitions of left and right minimal polynomials. Furthermore, since $P_{\Lambda,\ell}^{e_\ell}(\alpha_i) = 0$, we have, by (4.3) and (5.16),

$$L_{\alpha_i} \tilde{f}_{\mathbf{r},s} = (L_{\alpha_i} P_{\Lambda,\ell}) \cdot \gamma_s \cdot P_{\Omega_s,\mathbf{r}}$$

and since $P_{\Omega_s,\mathbf{r}}^{e_r}(\beta_i) = 0$ for all $i = 1, \dots, k$, equalities (5.18) hold.

It is clear from (5.16) that $\deg(\tilde{f}_{\mathbf{r},s}) = \deg(P_{\Lambda,\ell}) + \deg(P_{\Omega_j,\mathbf{r}}) = n + m - 1$ (since $\gamma_s \neq 0$). If f is any polynomial subject to conditions (5.17), (5.18), then the polynomial $f - \tilde{f}_{\mathbf{r},s}$ belongs to $P_{\Lambda,\ell} \cdot \mathbb{H}[z] \cdot P_{\Omega,\mathbf{r}}$ (by Theorem 5.1) and therefore, either $f \equiv \tilde{f}_{\mathbf{r},s}$ or $\deg(f - \tilde{f}_{\mathbf{r},s}) \geq \deg(P_{\Lambda,\ell}) + \deg(P_{\Omega,\mathbf{r}}) = m + n$. This implies the uniqueness of a low-degree solution. \square

Lemma 5.5. *Under assumptions (5.1), let , let $s \in \{k+1, \dots, n\}$ and let*

$$\rho_s = \begin{cases} P_{\Lambda_s,\ell}^{e_\ell}(\alpha_s)^{-1} \cdot (P_{\Omega,\mathbf{r}}^\sharp P_{\Omega,\mathbf{r}})(\alpha_s)^{-1} \cdot c_s \cdot P_{\Omega,\mathbf{r}}^{\sharp e_\ell}(c_s^{-1} \alpha_s c_s), & \text{if } c_s \neq 0, \\ 0, & \text{if } c_s = 0. \end{cases} \quad (5.20)$$

Then

$$\tilde{f}_{\ell,s}(z) = P_{\Lambda_s,\ell}(z) \cdot \rho_s \cdot P_{\Omega,\mathbf{r}}(z) \quad (5.21)$$

is a unique polynomial of degree less than $n + m$ satisfying conditions (5.18) and

$$f^{e_\ell}(\alpha_s) = c_s, \quad f^{e_\ell}(\alpha_i) = 0 \quad (i \neq s), \quad f^{e_r}(\beta_j) = d_j \quad (j = 1, \dots, m). \quad (5.22)$$

Proof. The proof is similar to that of Lemma 5.4. If $c_s = 0$, the statement follows from Theorem 5.1. If $c_s \neq 0$, the formula (5.20) makes sense, since $P_{\Lambda_s,\ell}(\alpha_s) \neq 0$ and since $\alpha_s \notin \bigcup_{j=1}^m [\beta_j] = \mathcal{Z}(P_{\Omega,\mathbf{r}}^\sharp P_{\Omega,\mathbf{r}})$. If $c_s \neq 0$, it is seen from (5.21) that $\deg(\tilde{f}_{\ell,s}) = \deg(P_{\Lambda_s,\ell}) + \deg(P_{\Omega,\mathbf{r}}) = n - 1 + m$, while the uniqueness of a low-degree solution follows from Theorem 5.1 (as in the proof of Lemma 5.4). It remains to show that $\tilde{f}_{\ell,s}$ indeed satisfies conditions (5.18) and (5.22).

Since $P_{\Omega,\mathbf{r}}^{e_r}(\beta_j) = 0$ for $1 \leq j \leq m$, we have for the right backward shift R_{β_j} , (4.2),

$$R_{\beta_j} \tilde{f}_{\ell,s} = R_{\beta_j} (P_{\Lambda_s,\ell} \cdot \rho_s \cdot P_{\Omega,\mathbf{r}}) = P_{\Lambda_s,\ell} \cdot \rho_s \cdot (R_{\beta_j} P_{\Omega,\mathbf{r}}).$$

Since $P_{\Lambda_s,\ell}(\alpha_i) = 0$ for $i = 1, \dots, k$ (recall that $s > k$), we have in particular, $(R_{\beta_i} \tilde{f}_{\ell,s})^{e_\ell}(\alpha_i) = 0$ for $i = 1, \dots, k$, and the latter equalities are equivalent to (5.18) due to (5.3).

Furthermore, if $c_s \neq 0$, the formula (5.20) can be written equivalently as

$$P_{\Lambda_s,\ell}^{e_\ell}(\alpha_s) \cdot \rho_s = \left[(P_{\Omega,\mathbf{r}}^\sharp P_{\Omega,\mathbf{r}})(\alpha_s) \right]^{-1} \cdot c_s \cdot P_{\Omega,\mathbf{r}}^{\sharp e_\ell}(c_s^{-1} \alpha_s c_s)$$

and then by implication \Leftarrow in (5.14) (with $\delta = P_{\Lambda_s,\ell}^{e_\ell}(\alpha_s) \cdot \rho_s$), we have

$$c_s = P_{\Lambda_s,\ell}^{e_\ell}(\alpha_s) \cdot \rho_s \cdot P_{\Omega,\mathbf{r}}^{e_\ell} \left(\rho_s^{-1} \cdot P_{\Lambda_s,\ell}^{e_\ell}(\alpha_s)^{-1} \cdot \alpha_s \cdot P_{\Lambda_s,\ell}^{e_\ell}(\alpha_s) \cdot \rho_s \right).$$

On the other hand, formula (2.6) applied to the product (5.21) gives

$$\tilde{f}_{\ell,s}^{e\ell}(\alpha_s) = P_{\Lambda_s,\ell}^{e\ell}(\alpha_s) \cdot \rho_s \cdot P_{\Omega_s,\mathbf{r}}^{e\ell} \left(\rho_s^{-1} \cdot P_{\Lambda_s,\ell}^{e\ell}(\alpha_s)^{-1} \cdot \alpha_s \cdot P_{\Lambda_s,\ell}^{e\ell}(\alpha_s) \cdot \rho_s \right),$$

and the two latter equalities imply $\tilde{f}_{\ell,i}^{e\ell}(\alpha_s) = c_s$. Verification of all other equalities in (5.22) is the same as in the proof of Lemma 5.4. \square

Lemma 5.6. *Under assumptions (5.1) and (5.2), let $s \in \{1, \dots, k\}$, let $\tilde{\alpha}_s$ and $\tilde{\beta}_s$ be defined as in (5.10) (so that $\alpha_s, \beta_s, \tilde{\alpha}_s, \tilde{\beta}_s$ belong to the same conjugacy class) and let $\Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$ be the plane defined via formula (4.10). Then*

(1) *Quaternions γ_s and ρ_s given by (compare with (5.15) and (5.20))*

$$\gamma_s = \begin{cases} P_{\Lambda_s,\ell}^{\#e\mathbf{r}}(d_s \beta_s d_s^{-1}) \cdot d_s \cdot (P_{\Lambda_s,\ell}^{\#} P_{\Omega_s,\mathbf{r}})(\beta_s)^{-1} \cdot P_{\Omega_s,\mathbf{r}}^{e\mathbf{r}}(\beta_s)^{-1}, & \text{if } d_s \neq 0, \\ 0, & \text{if } d_s = 0, \end{cases} \quad (5.23)$$

$$\rho_s = \begin{cases} P_{\Lambda_s,\ell}^{e\ell}(\alpha_s)^{-1} \cdot (P_{\Omega_s,\mathbf{r}}^{\#} P_{\Omega_s,\mathbf{r}})(\alpha_s)^{-1} \cdot c_s \cdot P_{\Omega_s,\mathbf{r}}^{\#e\ell}(c_s^{-1} \alpha_s c_s), & \text{if } c_s \neq 0, \\ 0, & \text{if } c_s = 0. \end{cases} \quad (5.24)$$

are subject to equality

$$\overline{\tilde{\alpha}_s}(\rho_s - \gamma_s) = (\rho_s - \gamma_s)\tilde{\beta}_s. \quad (5.25)$$

(2) *All polynomials f of degree less than $n + m$ satisfying conditions*

$$f^{e\ell}(\alpha_s) = c_s, \quad f^{e\ell}(\alpha_j) = 0 \quad (j \in \{1, \dots, n\} \setminus \{s\}), \quad (5.26)$$

$$f^{e\mathbf{r}}(\beta_s) = d_s, \quad f^{e\mathbf{r}}(\beta_j) = 0 \quad (j \in \{1, \dots, m\} \setminus \{s\}), \quad (5.27)$$

$$(L_{\alpha_i} f)^{e\mathbf{r}}(\beta_i) = 0 \quad (i \in \{1, \dots, k\} \setminus \{s\}), \quad (5.28)$$

are given by the formula

$$f(z) = \tilde{f}_s(z) + P_{\Lambda_s,\ell}(z) \cdot \mu_s \cdot P_{\Omega_s,\mathbf{r}}(z) \quad (5.29)$$

where $\mu_s \in \Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$ is a free parameter and where

$$\tilde{f}_s(z) = P_{\Lambda_s,\ell}(z) \cdot \rho_s \cdot P_{\Omega_s,\mathbf{r}}(z) + P_{\Lambda_s,\ell}(z) \cdot (2\text{Im}(\tilde{\alpha}_s))^{-1}(\rho_s - \gamma_s) \cdot P_{\Omega_s,\mathbf{r}}(z). \quad (5.30)$$

Proof. We first observe that d_s and c_s are recovered from (5.23) and (5.24) by

$$d_s = (P_{\Lambda_s,\ell} \cdot \gamma_s \cdot P_{\Omega_s,\mathbf{r}})^{e\mathbf{r}}(\beta_s), \quad c_s = (P_{\Lambda_s,\ell} \cdot \rho_s \cdot P_{\Omega_s,\mathbf{r}})^{e\ell}(\alpha_s). \quad (5.31)$$

The trivial cases where $d_s = c_s = 0$ are clear. If $d_s \neq 0$, we have from (5.23),

$$\gamma_s \cdot P_{\Omega_s,\mathbf{r}}^{e\mathbf{r}}(\beta_s) = P_{\Lambda_s,\ell}^{\#e\mathbf{r}}(d_s \beta_s d_s^{-1}) \cdot d_s \cdot (P_{\Lambda_s,\ell}^{\#} P_{\Omega_s,\mathbf{r}})(\beta_s)^{-1}$$

and by implication \Leftarrow in (5.13) and formula (2.7) we conclude

$$\begin{aligned} d_s &= P_{\Lambda_s,\ell}^{e\mathbf{r}} \left(\gamma_s \cdot P_{\Omega_s,\mathbf{r}}^{e\mathbf{r}}(\beta_s) \cdot \beta_s \cdot P_{\Omega_s,\mathbf{r}}^{e\mathbf{r}}(\beta_s)^{-1} \cdot \gamma_s^{-1} \right) \cdot \gamma_s \cdot P_{\Omega_s,\mathbf{r}}^{e\mathbf{r}}(\beta_s) \\ &= (P_{\Lambda_s,\ell} \cdot \gamma_s \cdot P_{\Omega_s,\mathbf{r}})^{e\mathbf{r}}(\beta_s), \end{aligned}$$

which confirms the first equality in (5.31). The second equality for $c_s \neq 0$ is verified in much the same way. Let us introduce

$$\tilde{d}_s := (P_{\Lambda_s,\ell} \cdot \rho_s \cdot P_{\Omega_s,\mathbf{r}})^{e\mathbf{r}}(\beta_s). \quad (5.32)$$

It follows from (5.31) and (5.32) that c_s and \tilde{d}_s are left and right values of the same polynomial at conjugate points α_s and β_s ; therefore

$$\overline{\alpha}_s(c_s - \tilde{d}_s) = (c_s - \tilde{d}_s)\beta_s,$$

by Corollary 4.4. On the other hand, by the assumption (5.2),

$$\overline{\alpha}_s(c_s - d_s) = (c_s - d_s)\beta_s. \quad (5.33)$$

Combining the two latter equalities gives $\overline{\alpha}_s(\tilde{d}_s - d_s) = (\tilde{d}_s - d_s)\beta_s$. Substituting the formulas (5.31) and (5.32) for d_s and \tilde{d}_s into the latter equality and taking into account that right evaluation is linear on $\mathbb{H}[z]$, we get

$$\overline{\alpha}_s \cdot (P_{\Lambda_s, \ell} \cdot (\rho_s - \gamma_s) \cdot P_{\Omega_s, \mathbf{r}})^{e_r}(\beta_s) = (P_{\Lambda_s, \ell} \cdot (\rho_s - \gamma_s) \cdot P_{\Omega_s, \mathbf{r}})^{e_r}(\beta_s) \cdot \beta_s$$

which can be equivalently written as

$$((z - \overline{\alpha}_s) \cdot P_{\Lambda_s, \ell} \cdot (\rho_s - \gamma_s) \cdot P_{\Omega_s, \mathbf{r}})^{e_r}(\beta_s) = 0.$$

Using formula (2.7) and definition (5.10) of $\tilde{\beta}_s$, we write the latter formula as

$$((z - \overline{\alpha}_s) \cdot P_{\Lambda_s, \ell} \cdot (\rho_s - \gamma_s))^{e_r}(\tilde{\beta}_s) \cdot P_{\Omega_s, \mathbf{r}}^{e_r}(\beta_s) = 0$$

and since $P_{\Omega_s, \mathbf{r}}^{e_r}(\beta_s) \neq 0$, we have

$$((z - \overline{\alpha}_s) \cdot P_{\Lambda_s, \ell} \cdot (\rho_s - \gamma_s))^{e_r}(\tilde{\beta}_s) = 0. \quad (5.34)$$

Observe that if $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ is permuted by moving α_s to the rightmost spot, then the recursion (2.13) produces $P_{\Lambda, \ell}$ in the form

$$P_{\Lambda, \ell}(z) = P_{\Lambda_s, \ell}(z) \cdot \left(z - P_{\Lambda_s, \ell}^{e_\ell}(\alpha_s)^{-1} \alpha_s P_{\Lambda_s, \ell}^{e_\ell}(\alpha_s) \right) = P_{\Lambda_s, \ell}(z) \cdot (z - \tilde{\alpha}_s). \quad (5.35)$$

Since $P_{\Lambda, \ell}$ has only one left zero in the conjugacy class $[\alpha_s]$, we have

$$\begin{aligned} (z - \overline{\alpha}_s) \cdot P_{\Lambda, \ell}(z) &= \mathcal{X}_{[\alpha_s]}(z) \cdot (L_{\alpha_s} P_{\Lambda, \ell})(z) \\ &= (L_{\alpha_s} P_{\Lambda, \ell})(z) \cdot \mathcal{X}_{[\alpha_s]}(z) \\ &= (L_{\alpha_s} P_{\Lambda, \ell})(z) \cdot \mathcal{X}_{[\tilde{\alpha}_s]}(z) = (L_{\alpha_s} P_{\Lambda, \ell})(z) \cdot (z - \overline{\tilde{\alpha}}_s)(z - \tilde{\alpha}_s) \end{aligned}$$

which being compared with (5.35) implies

$$(L_{\alpha_s} P_{\Lambda, \ell})(z) \cdot (z - \overline{\tilde{\alpha}}_s) = (z - \overline{\alpha}_s) \cdot P_{\Lambda_s, \ell}(z). \quad (5.36)$$

Substituting this identity into (5.34) gives

$$\left((L_{\alpha_s} P_{\Lambda, \ell}) \cdot (z - \overline{\tilde{\alpha}}_s) \cdot (\rho_s - \gamma_s) \right)^{e_r}(\tilde{\beta}_s) = 0.$$

Since the polynomial $L_{\alpha_s} P_{\Lambda, \ell}$ has no zeros in the conjugacy class $[\alpha_s] = [\tilde{\beta}_s]$, the latter equality implies

$$\left((z - \overline{\tilde{\alpha}}_s) \cdot (\rho_s - \gamma_s) \right)^{e_r}(\tilde{\beta}_s) = 0$$

which is the same as (5.25). This completes the proof of the first statement of the lemma.

To prove the second statement, we first exclude the first condition in (5.27) and consider the problem with the remaining interpolation conditions in (5.26)–(5.28). In

this reduced setting, the role of Ω is played by the set Ω_s (the point β_s is temporarily excluded from Ω) and, since $\alpha_1, \dots, \alpha_n$ does not have conjugates in Ω_s , the reduced problem is of the type considered in Lemma 5.5. Combining Lemma 5.5 and Theorem 5.1 we conclude that all solutions of the reduced problem are given by the formula

$$f(z) = P_{\Lambda_s, \ell}(z) \cdot \rho_s \cdot P_{\Omega_s, \mathbf{r}}(z) + P_{\Lambda, \ell}(z) \cdot g(z) \cdot P_{\Omega_s, \mathbf{r}}(z) \quad (5.37)$$

for some $g \in \mathbb{H}[z]$ (by Theorem 5.1, the second term on the right side of (5.37) is a general solution to the reduced homogeneous problem). Note that ρ_s in (5.37) is defined by formula (5.20) but with Ω replaced by Ω_s , that is, by formula (5.24). Let us observe the factorization

$$P_{\Omega, \mathbf{r}}(z) = (z - \tilde{\beta}_s) \cdot P_{\Omega_s, \mathbf{r}}(z) \quad (5.38)$$

! which follows from the recursion (2.14) applied to the set $\Omega = \{\beta_1, \dots, \beta_m\}$ permuted by moving β_s to the rightmost spot (the right-sided counterpart of (5.35)). Furthermore, let us take g in the form $g(z) = \psi_s + h(z) \cdot (z - \tilde{\beta}_s)$ where $\psi_s = g^{e_r}(\beta_s)$ and $h = R_{\tilde{\beta}_s} g$. Substituting this representation into (5.37) and taking into account (5.38) we get

$$f(z) = P_{\Lambda_s, \ell}(z) \cdot \rho_s \cdot P_{\Omega_s, \mathbf{r}}(z) + P_{\Lambda, \ell}(z) \cdot \psi_s \cdot P_{\Omega_s, \mathbf{r}}(z) + P_{\Lambda_s, \ell}(z) \cdot h(z) \cdot P_{\Omega, \mathbf{r}}(z).$$

The rightmost term on the right represents the general solution to the homogeneous problem (5.3), (5.6) and is of degree at least $m + n$ if $h \not\equiv 0$. Let us focus on the low-degree part corresponding to the choice of $h \equiv 0$. The main objective now is to specify ψ_s in such a way that the function

$$f = P_{\Lambda_s, \ell} \cdot \rho_s \cdot P_{\Omega_s, \mathbf{r}} + P_{\Lambda, \ell} \cdot \psi_s \cdot P_{\Omega_s, \mathbf{r}} \quad (5.39)$$

will satisfy the first condition in (5.27). Making use of factorizations (5.35), (5.38) and setting

$$\Phi_s := \rho_s + \psi_s \tilde{\beta}_s - \tilde{\alpha}_s \psi_s, \quad (5.40)$$

we rewrite (5.39)

$$\begin{aligned} f(z) &= P_{\Lambda_s, \ell}(z) \cdot \rho_s \cdot P_{\Omega_s, \mathbf{r}}(z) + P_{\Lambda_s, \ell}(z) \cdot (z - \tilde{\alpha}_s) \cdot \psi_s \cdot P_{\Omega_s, \mathbf{r}}(z) \\ &= P_{\Lambda_s, \ell}(z) \cdot \Phi_s \cdot P_{\Omega_s, \mathbf{r}}(z) + P_{\Lambda_s, \ell}(z) \cdot \psi_s \cdot (z - \tilde{\beta}_s) \cdot P_{\Omega_s, \mathbf{r}}(z) \\ &= P_{\Lambda_s, \ell}(z) \cdot \Phi_s \cdot P_{\Omega_s, \mathbf{r}}(z) + P_{\Lambda_s, \ell}(z) \cdot \psi_s \cdot P_{\Omega, \mathbf{r}}(z). \end{aligned} \quad (5.41)$$

Since $P_{\Omega, \mathbf{r}}^{e_r}(\beta_s) = 0$, we conclude that f of the form (5.41) satisfies condition $f^{e_r}(\beta_s) = d_s$ if and only if

$$(P_{\Lambda_s, \ell} \cdot \Phi_s \cdot P_{\Omega_s, \mathbf{r}})^{e_r}(\beta_s) = d_s. \quad (5.42)$$

Since the conjugacy class $[\beta_s] = [\alpha_s]$ is disjoint with $\mathcal{Z}_{\mathbf{r}}(P_{\Lambda_s, \ell})$ and $\mathcal{Z}_{\mathbf{r}}(P_{\Omega_s, \mathbf{r}})$, it follows from (5.42) and (5.23) that $\Phi_s = 0 \Leftrightarrow d_s = 0 \Leftrightarrow \gamma_s = 0$. If $d_s \neq 0$, then Φ_s is uniquely recovered from (5.42) as well as γ_s is recovered from the first formula in (5.31). We then conclude from (5.31) and (5.42) that $\Phi_s = \gamma_s$. We summarize: f satisfies conditions (5.26)–(5.28) and is of degree less than $n + m$ if and only if it is of the form (5.39) where ψ_s satisfies (5.40), i.e. (since $\Phi_s = \gamma_s$), if and only if ψ_s is a solution to the Sylvester equation

$$\tilde{\alpha}_s \psi_s - \psi_s \tilde{\beta}_s = \rho_s - \gamma_s. \quad (5.43)$$

The latter equation is consistent, by Lemma 4.3 and due to equality (5.25). By Lemma 4.3, all solutions ψ_s to the equation (5.43) are given by the formula

$$\psi_s = (2\text{Im}(\tilde{\alpha}_s))^{-1}(\rho_s - \gamma_s) + \mu_s, \quad \mu_s \in \Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$$

which being substituted into (5.39), gives (5.28) completing the proof. \square

Remark 5.7. The second term on the right side of (5.29) looks asymmetric with respect to the sets Λ and Ω . However, since the membership $\mu_s \in \Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$ means that $\tilde{\alpha}_s \mu_s = \mu_s \tilde{\beta}_s$ (by Lemma 4.3), we have $(z - \tilde{\alpha}_s) \cdot \mu_s = \mu_s \cdot (z - \tilde{\beta}_s)$, and the alternative representation

$$P_{\Lambda, \ell} \cdot \mu_s \cdot P_{\Omega, \mathbf{r}} = P_{\Lambda, \ell} \cdot (z - \tilde{\alpha}_s) \cdot \mu_s \cdot P_{\Omega, \mathbf{r}} = P_{\Lambda, \ell} \cdot \mu_s \cdot (z - \tilde{\beta}_s) \cdot P_{\Omega, \mathbf{r}} = P_{\Lambda, \ell} \cdot \mu_s \cdot P_{\Omega, \mathbf{r}}$$

follows from (5.35) and (5.40).

The reason for non-uniqueness of a low-degree solution to the problem (5.26)–(5.28) is that the value of $(L_{\alpha_s} f)^{er}(\beta_s)$ is not fixed.

Remark 5.8. For each $q_s \in \mathbb{H}$ satisfynig the Sylvester equality

$$\alpha_s q - q \beta_s = c_s - d_s, \quad (5.44)$$

there exists a unique polynomial f of the form (5.29) such that $(L_{\alpha_s} f)^{er}(\beta_s) = q_s$.

Proof. For any $\mu_s \in \Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$ and f of the form (5.29), we have

$$(L_{\alpha_s} f)^{er}(\beta_s) = (L_{\alpha_s} \tilde{f}_s)^{er}(\beta_s) + ((L_{\alpha_s} P_{\Lambda, \ell}) \cdot \mu_s \cdot P_{\Omega, \mathbf{r}})^{er}(\beta_s), \quad (5.45)$$

and the quaternion $q_s := (L_{\alpha_s} f)^{er}(\beta_s)$ satisfies equality (5.44), by Remark 4.1. On the other hand, for any $\tilde{q}_s \in \mathbb{H}$, the equation

$$((L_{\alpha_s} P_{\Lambda, \ell}) \cdot \mu_s \cdot P_{\Omega, \mathbf{r}})^{er}(\beta_s) = \tilde{q}_s$$

can be solved for μ_s (using the implication \Rightarrow in (5.13)) as follows:

$$\mu_s = (L_{\alpha_s} P_{\Lambda, \ell})^{\sharp er}(\tilde{q}_s \beta_s \tilde{q}_s^{-1}) \cdot d_s \cdot (P_{\Lambda, \ell}^{\sharp}(P_{\Lambda, \ell})(\beta_s))^{-1} \cdot P_{\Omega, \mathbf{r}}^{er}(\beta_s)^{-1}, \quad \text{if } \tilde{q}_s \neq 0, \quad (5.46)$$

and $\mu_s = 0$ if $\tilde{q}_s = 0$; in fact, we should have used $L_{\alpha_s} P_{\Lambda, \ell}$ rather than $P_{\Lambda, \ell}$ in the latter formula but, although these polynomials are distinct in general, it follows from identity (5.36) and the third equality in (2.3) that $(L_{\alpha_s} P_{\Lambda, \ell})^{\sharp}(L_{\alpha_s} P_{\Lambda, \ell}) = P_{\Lambda, \ell}^{\sharp} P_{\Lambda, \ell}$. We now conclude from (5.45) that f of the form (5.29) satisfies equality $(L_{\alpha_s} f)^{er}(\beta_s) = q_s$ if and only if the parameter μ_s is defined by formula (5.46) with $\tilde{q}_s = q_s - (L_{\alpha_s} \tilde{f}_s)^{er}(\beta_s)$. \square

The next theorem is the main result of the paper.

Theorem 5.9. Under assumptions (5.1) and (5.2), let $\tilde{\alpha}_s$ and $\tilde{\beta}_s$ be defined as in (5.10), let $\Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$ be the plane defined via formula (4.10) and let polynomials $\tilde{f}_{\mathbf{r}, j}$ ($k < j \leq m$), $f_{\ell, i}$ ($k < i \leq n$) and \tilde{f}_s ($1 \leq s \leq k$) be defined via formulas (5.16),

(5.21) and (5.30), respectively. Then all polynomials $f \in \mathbb{H}[z]$ satisfying conditions (1.7), (1.8) are given by the formula

$$\begin{aligned} f(z) = & \sum_{i=k+1}^n \tilde{f}_{\ell,i}(z) + \sum_{j=k+1}^m \tilde{f}_{\mathbf{r},j}(z) + \sum_{s=1}^k \tilde{f}_s(z) + \sum_{s=1}^k P_{\Lambda,s,\ell}(z) \cdot \mu_s \cdot P_{\Omega,\mathbf{r}}(z) \\ & + P_{\Lambda,\ell}(z) \cdot h(z) \cdot P_{\Omega,\mathbf{r}}(z), \end{aligned} \quad (5.47)$$

where $\mu_s \in \Pi_{\tilde{\alpha}_s, \tilde{\beta}_s}$ and $h \in \mathbb{H}[z]$ are free parameters.

Proof. The “if” part follows immediately from Theorem 5.1, and Lemmas 5.4, 5.5, 5.6. Now let f be any polynomial satisfying conditions (1.7), (1.8). For each $s \in \{1, \dots, k\}$, define μ_s a unique element $\mu_s \in \mathbb{H}$ (as explained in Remark 5.8) such that

$$(L_{\alpha_s}(P_{\Lambda,\ell} \cdot \mu_s \cdot P_{\Omega_s,\mathbf{r}}))^{\mathbf{er}}(\beta_s) = (L_{\alpha_s}f)^{\mathbf{er}}(\beta_s) - (L_{\alpha_s}\tilde{f}_s)^{\mathbf{er}}(\beta_s).$$

Then the polynomial $f - \sum_{i=k+1}^n \tilde{f}_{\ell,i} - \sum_{j=k+1}^m \tilde{f}_{\mathbf{r},j} - \sum_{s=1}^k \tilde{f}_s - \sum_{s=1}^k P_{\Lambda,s,\ell} \cdot \mu_s \cdot P_{\Omega,\mathbf{r}}$ solves the homogeneous problem (5.3), (5.6) and hence, belongs to $P_{\Lambda,\ell} \cdot \mathbb{H}[z] \cdot P_{\Omega_s,\mathbf{r}}$ from which representation (5.47) follows. \square

Combining Theorem 5.9 and Remark 5.8 one can get the non-homogeneous version of Theorem 5.1.

Theorem 5.10. *Under assumptions (5.1) and (5.2), let q_s be a solution to the Sylvester equation (5.44) for $s = 1, \dots, k$. The set of all $f \in \mathbb{H}[z]$ satisfying conditions (1.7), (1.8) and*

$$(L_{\alpha_s}f)^{\mathbf{er}}(\beta_s) = q_s \quad \text{for } s = 1, \dots, k$$

is parametrized by formula (5.47) with free parameter $h \in \mathbb{H}[z]$ and the elements μ_s uniquely determined by q_s (as explained in Remark 5.8).

Proof of Theorem 5.2: We let $c_i = d_j = 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$ and then conclude from formulas (5.15), (5.20), (5.23), (5.24) that $\rho_i = \gamma_j = 0$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Then it follows from formulas (5.16), (5.21), (5.30) that all elementary polynomials $\tilde{f}_{\ell,i}$, $\tilde{f}_{\mathbf{r},j}$, \tilde{f}_s are zero polynomials. Now description (5.11) in Theorem 5.2 follows from (5.47). \square

Another particular case of Theorem 5.9 (where $k = 0$) admits a fairly simple answer.

Theorem 5.11. *Assume that $[\alpha_i] \cap \Omega = \emptyset$ ($1 \leq i \leq n$) and $[\beta_j] \cap \Lambda = \emptyset$ ($1 \leq j \leq m$). Then all polynomials $f \in \mathbb{H}[z]$ satisfying conditions (1.7), (1.8) are given by*

$$f(z) = \sum_{i=1}^n \tilde{f}_{\ell,i}(z) + \sum_{j=1}^m \tilde{f}_{\mathbf{r},j}(z) + P_{\Lambda,\ell}(z) \cdot h(z) \cdot P_{\Omega,\mathbf{r}}(z), \quad h \in \mathbb{H}[z] \quad (5.48)$$

where $\tilde{f}_{\mathbf{r},j}$ and $\tilde{f}_{\ell,i}$ are defined via formulas (5.16), (5.21). The two first terms on the right side of (5.48) present a unique polynomial of degree less than $m + n$ satisfying conditions (1.7), (1.8).

Specializing formula (5.48) further to the case where $\Omega = \emptyset$ and therefore, $P_{\Omega, \mathbf{r}} \equiv 1$ and $\rho_i = P_{\Lambda_i, \ell}^{e_\ell}(\alpha_i)^{-1}$ (according to (5.20)), recovers Theorem 3.2. Letting $\Lambda = \emptyset$ in Theorem 5.11 and making appropriate adjustments we recover Theorem 3.3.

6. ALTERNATIVE FORMULAS FOR LOW-DEGREE PARTICULAR SOLUTIONS

As in the complex case, a low-degree solution (in case it is unique) can be constructed via several different schemes. Although the produced formulas are not as explicit in terms of interpolation data as the Lagrange's formula (1.2), the algorithms might be more efficient from the computational point of view. For the left-sided problem (1.7), one can pick any basis $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ for the space of polynomials of degree less than n and find f_ℓ in the form $f_\ell(z) = \sum \mathbf{a}_j(z)\varphi_j$ with the coefficients φ_i obtained from the linear system

$$\sum_{j=1}^n \mathbf{a}_j^{e_\ell}(\alpha_i)\varphi_j = c_i \quad \text{for } i = 1, \dots, n. \quad (6.1)$$

Theorem 5.1 implies in particular, that the latter system has a unique solution for any choice of c_1, \dots, c_n . This, in turn, implies that the matrix A of the system (6.1) is invertible and therefore, the left Lagrange polynomial can be written as

$$\tilde{f}_\ell(z) = [\mathbf{a}_1(z) \ \dots \ \mathbf{a}_n(z)] A^{-1}C, \quad A = [\mathbf{a}_j^{e_\ell}(\alpha_i)]_{i,j=1}^n, \quad C = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}. \quad (6.2)$$

We mention three “canonical” bases. If we let $\mathbf{a}_j = P_{\Lambda_j, \ell}$ to be the **lmp** of the set $\Lambda_j := \Lambda \setminus \{\alpha_j\}$, the matrix A is diagonal and the formula (6.2) amounts to the Lagrange formula (3.8). The monomial basis $\mathbf{a}_j(z) = z^{j-1}$ ($j = 1, \dots, n$) leads to the Vandermonde matrix $A = [\alpha_i^{j-1}]_{i,j=1}^n$. The third convenient basis $\mathbf{a}_j = p_{j-1}$ ($j = 1, \dots, n$) is suggested by the recursion (2.13). Recall that $p_0 \equiv 1$ and $p_j(z)$ is the **lmp** for the set $\{\alpha_1, \dots, \alpha_j\}$. Therefore, the matrix A is low triangular:

$$A = [p_{k-1}^{e_\ell}(\alpha_i)]_{i,k=1}^n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & p_1^{e_\ell}(\alpha_2) & 0 & \dots & 0 \\ 1 & p_1^{e_\ell}(\alpha_3) & p_2^{e_\ell}(\alpha_3) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_1^{e_\ell}(\alpha_n) & p_2^{e_\ell}(\alpha_n) & \dots & p_{n-1}^{e_\ell}(\alpha_n) \end{bmatrix},$$

and the formula (6.2) now amounts to the Newton's interpolation formula

$$\tilde{f}_\ell(z) = \varphi_0 + p_1(z)\varphi_1 + \dots + p_{n-1}(z)\varphi_{n-1}, \quad (6.3)$$

where the coefficients φ_j are recursively recovered from the triangular system

$$\sum_{k=0}^j p_k^{e_\ell}(\alpha_{j+1})\varphi_k = c_j \quad (j = 0, \dots, n-1). \quad (6.4)$$

Due to the triangular structure, the Newton's scheme easily incorporates additional interpolation nodes: to get the formula for the Lagrange polynomial satisfying the

additional condition $f(\alpha_{n+1}) = c_{n+1}$ (assuming that α_{n+1} is equivalent to at most one point from $\{\alpha_1, \dots, \alpha_n\}$), it suffices to calculate $p_n(z)$, to find φ_n from (6.4) and then to add the extra term $p_n(z)\varphi_n$ to the expression on the right side of (6.3). To apply the Lagrange formula (3.8) in a similar situation, one need to recalculate all basis polynomials $P_{\Lambda_i, \ell}$ for $i = 1, \dots, n+1$.

If the basis polynomials are such that $\deg \mathbf{a}_j = j-1$ for $j = 1, \dots, n$, then the degree of the Lagrange polynomial \tilde{f}_ℓ can be determined from interpolation data as the minimal integer n_0 such that the column C of target values (6.2) belongs to the right linear span of n_0 leftmost columns of the matrix A . For example, for the monomial basis $\mathbf{a}_j = p_{j-1}$, we have

$$\deg(\tilde{f}_\ell) = \min \left\{ k : \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \text{span}_{\mathbf{r}} \left\{ \begin{bmatrix} \alpha_1^{j-1} \\ \vdots \\ \alpha_n^{j-1} \end{bmatrix}, j = 1, \dots, k \right\} \right\}. \quad (6.5)$$

Similar observations hold true for the right-sided sided problem (1.8) and can be applied to the two-sided problem as follows. Let us assume for the sake of simplicity that the low-degree solution is unique, i.e., that is none of the left nodes is equivalent to a right node. By Theorem 3.2, the low degree solution must be of the form

$$\tilde{f} = \tilde{f}_\ell + P_{\Lambda, \ell} h \quad (6.6)$$

for some $h \in \mathbb{H}[z]$ of degree less than m . Formula (6.6) guarantees that \tilde{f} satisfies left conditions (1.7). Let us introduce the elements

$$\tilde{d}_j := d_j - \tilde{f}_\ell^{er}(\beta_j) \quad \text{for } j = 1, \dots, m.$$

Using the equivalence (5.13), it can be shown that \tilde{f} of the form (6.6) satisfies conditions (1.8) if and only if the parameter h is subject to right conditions

$$h^{er}(\beta_j) = \delta_j := \begin{cases} P_{\Lambda, \ell}^{\sharp er}(\tilde{d}_j \beta_j \tilde{d}_j^{-1}) \cdot \tilde{d}_j \cdot (P_{\Lambda, \ell}^{\sharp} P_{\Lambda, \ell})(\beta_j)^{-1}, & \text{if } \tilde{d}_j \neq 0, \\ 0 & \text{if } \tilde{d}_j = 0, \end{cases} \quad (6.7)$$

for $j = 1, \dots, m$. The unique h with $\deg(h) < m$ and satisfying conditions (6.7) can be written as

$$h(z) = DB^{-1} \begin{bmatrix} \mathbf{b}_1(z) \\ \vdots \\ \mathbf{b}_m(z) \end{bmatrix}, \quad B = [\mathbf{b}_i^{er}(\beta_j)]_{i,j=1}^n, \quad D = [\delta_1 \ \dots \ \delta_m] \quad (6.8)$$

for a fixed basis $\{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ of the space of polynomials of degree less than m . To construct h , we may use the monomial basis and the Vandermonde matrix $B = [\beta_j^{i-1}]_{i,j=1}^m$. Alternatively, we may use the right version of the Newton's scheme with the upper triangular matrix $B = [q_{i-1}^{er}(\beta_k)]_{i,k=1}^n$ where the polynomials q_j are constructed recursively in (2.14). Similarly to (6.5), we have

$$\deg(h) = \min \{ s : [\delta_1 \ \dots \ \delta_m] \in \text{span}_{\ell} \{ [\beta_1^{i-1} \ \dots \ \beta_m^{i-1}], i = 1, \dots, s \} \}. \quad (6.9)$$

Substituting (6.8) into (6.6), we get the unique low degree solution \tilde{f} to the problem (1.7), (1.8). As we have shown, \tilde{f} can be constructed by applying the Newton's scheme first to the left problem and then to the recalculated right problem. It would be interesting to construct a "direct" triangular algorithm avoiding the recalculating step.

The value of $\deg(\tilde{f})$ in terms of interpolation data can be derived from (6.5), (6.6) and (6.9). Indeed, $\deg(\tilde{f}) = \deg(\tilde{f}_\ell)$ if $\tilde{d}_j = 0$ for all $j = 1, \dots, m$ (i.e., when it happens that \tilde{f}_ℓ accidentally satisfies all right conditions (1.8)). Otherwise, $\deg(\tilde{f}) = \deg(P_{\Lambda, \ell}) + \deg(h) = n + \deg(h)$ where the integer $\deg(h)$ is given in (6.9). Again, it would be interesting to express $\deg(\tilde{f})$ in terms of the original data rather than the elements δ_j 's. A more interesting question is to find minimal degree solutions (or at least the value of this minimal degree) in the general setting of Theorem 5.11. At the moment, it is not clear how small the degree of the polynomial in the top line of (5.47) can be done by varying μ_1, \dots, μ_k in the corresponding real planes of \mathbb{H} .

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